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Existence and uniqueness of an attractive nonlinear diffusion system

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ABSTRACT

We establish the existence and uniqueness of an attractive fractional coupled system. Such a system has applications in biological populations of cells. We confirm that the fractional system under consideration admits a global solution in the Sobolev space. The solution is shown to be unique. The technique is founded on analytic method of the fixed point theory and the fractional differential operator is scrutinized from the view of the Riemann–Liouville differential operator. Finally, we illustrate some entropy fractional differential inequalities regarding the solution of the system.

1. Introduction

Fractional calculus (including fractional order indefinite integral and differentiation) obtained by Leibniz was first applied for Abel's investigation of dynamical problems such as determining the curve γ for a descending particle along the curve γ with a given total time $f(t)$. This concept is utilized to analyze phenomena having singularities of type t^{α}. Fractional calculus and fractional order differentiation which is a nonlocal operator are used to study the effects of memories of Brownian motion (e.g. see Miller and Ross [1], or Oldham and Spanier [2], or Podlubny [3]). Moreover, fractional order differential equations have been successfully employed for modeling of many different processes and systems in physics, engineering, chemistry, biology, medicine and food processing (see [4–7]). In these applications, it is often important to consider boundary value problems such as existence and uniqueness of solutions for space–time fractional diffusion equations on bounded domains. The theory on existence and uniqueness of solutions of linear and nonlinear fractional differential equations has attracted the attention of many researchers (e.g see [8–15]). Here, we establish the existence and uniqueness of fractional differential system based on the Riemann–Liouville differential operator. We show that the proposed system implies a global solution in appropriate Sobolev spaces. The solution is considered to be unique and strong. We employ a method, based on analytic methods from the fixed point theory. Lastly, we illustrate some entropy fractional differential inequality regarding the solution of the system.

Fractional calculus originated from the Riemann–Liouville definition of fractional integral of order α in the form

$$_{a}I_{t}^{\alpha}f(t)=\int_{a}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau)d\tau.$$

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The fractional (arbitrary) order differential of the function f of order $\alpha > 0$ is given by

$$_aD_t^\alpha f(t) = \frac{d}{dt}\int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau)d\tau.$$

When $a = 0$, we shall denote $_0D_t^\alpha f(t) := D_t^\alpha f(t)$ and $_0I_t^\alpha f(t) := I_t^\alpha f(t)$ in the follow-up. From above, for $a = 0$, we conclude that

$$D_t^\alpha t^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} t^{m-\alpha}, \quad m > -1;\ 0 < \alpha < 1$$

and

$$I_t^\alpha t^m = \frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)} t^{m+\alpha}, \quad m > -1;\ \alpha > 0.$$

It was shown that the Leibniz rule for arbitrary differentiations of the smooth functions (their derivatives are continuous for all orders) $\phi(t)$ and $\psi(t)$, $t \in [a,b]$ can be formulated as follows (see [3]):

$$_aD_t^\alpha[\phi(t)\psi(t)] = \sum_{n=0}^{k}\binom{\alpha}{n} {_aD_t^{\alpha-n}}\phi(t)\, {_aD_t^n}\psi(t) - R_k^\alpha = \sum_{n=0}^{k}\binom{\alpha}{n} {_aD_t^{\alpha-n}}\psi(t)\, {_aD_t^n}\phi(t) - R_k^\alpha,$$

where $\alpha \leqslant k-1$,

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1-n)}$$

and R_k^α is the remainder of the series, which can be defined as follows:

$$R_k^\alpha = \left(\frac{1}{k!\Gamma(-\alpha)}\int_a^t (t-\tau)^{-\alpha-1}\phi(\tau)d\tau\right)\left(\int_\tau^t {_aD_t^{k+1}}\psi(\theta)(\tau-\theta)^k d\theta\right).$$

In addition, the fractional differential operator achieves linearity (see [3])

$$_aD_t^\alpha[\rho\phi(t) + \sigma\psi(t)] = \rho\, {_aD_t^\alpha}[\phi(t)] + \sigma\, {_aD_t^\alpha}[\psi(t)].$$

Recently, Alsaedi et. al. [16] presented an inequality for fractional derivatives related to the Leibniz rule as follows

Lemma 1.1. *Let one of the following conditions be satisfied*

- $u \in C([0,T])$ *and* $v \in C^\beta([0,T]), \alpha < \beta \leqslant 1$
- $v \in C([0,T])$ *and* $u \in C^\beta([0,T]), \alpha < \beta \leqslant 1$
- $u \in C^\beta([0,T])$ *and* $v \in C^\delta([0,T]), \alpha < \beta \leqslant \beta + \delta,\ \beta, \delta \in (0,1)$,

where $C^\gamma([0,T]) = \{u : [0,T] \to \mathbb{R} / |u(t) - u(t-h)| = O(h^\gamma)$ *uniformly for* $0 < t-h < t \leqslant T\}$. *Then we have*

$$D_t^\alpha(uv)(t) = u(t)D_t^\alpha v(t) + v(t)D_t^\alpha u(t) - \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{(u(s)-u(t))(v(s)-v(t))}{(t-s)^{\alpha+1}}ds - \frac{u(t)v(t)}{\Gamma(1-\alpha)t^{-\alpha}}$$

point-wise.

From Lemma 1.1, it follows that if u and v have the same sign and are both increasing or both decreasing, then

$$D_t^\alpha(uv)(t) \leqslant u(t)D_t^\alpha v(t) + v(t)D_t^\alpha u(t)$$

and for $u = v$, we obtain

$$D_t^\alpha(u^2)(t) \leqslant 2u(t)D_t^\alpha u(t). \tag{1}$$

2. Existence of the proposed system

We consider the dimensionless attraction system of fractional order

$$\begin{aligned}
D_t^\alpha u(t,x) &= \triangle u - \nabla.(\psi_1 u \nabla v) + \nabla.(\psi_2 u \nabla w), \quad x \in \Omega,\\
D_t^\alpha v(t,x) &= \triangle v + \phi_1 u - \phi_3 v, \quad x \in \Omega,\\
D_t^\alpha w(t,x) &= \triangle w + \phi_2 u - \phi_4 v, \quad x \in \Omega,\\
(u(x,0) &= u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \quad x \in \Omega),
\end{aligned} \tag{2}$$