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On a time fractional reaction diffusion equation

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ABSTRACT

A reaction diffusion equation with a Caputo fractional derivative in time and with various boundary conditions is considered. Under some conditions on the initial data, we show that solutions may experience blow-up in a finite time. However, for realistic initial conditions, solutions are global in time. Moreover, the asymptotic behavior of bounded solutions will be analyzed.

1. Introduction

In this paper, we consider the reaction diffusion system

$$ {}^cD^\alpha u - \Delta u = -u(1-u), \quad x \in \Omega,\ t > 0, \tag{1.1}$$

supplemented with:

– the homogeneous boundary condition

$$Bu = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.2}$$

where $Bu = u_{|\partial\Omega}$ or $Bu = \frac{\partial u}{\partial \nu}_{|\partial\Omega}$, ν the outward normal to $\partial\Omega$ (if $\Omega = \mathbb{R}^N$, the condition $Bu = 0$ is omitted but we add $\lim_{|x|\to\infty} u(x,t) = 0$),

- and with the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega; \tag{1.3}$$

the initial data $u_0(x)$ is a given positive and bounded function. Here $d > 0$ is the diffusion coefficient and ${}^cD^\alpha$ is the time fractional Caputo derivative of order $0 < \alpha < 1$ defined by

$$ {}^cD^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u}{\partial s}(x,s)(t-s)^{-\alpha}\, ds$$

for a function u differentiable in the time variable [4,10].

Our paper is motivated by the recent one of Nakagawa, Sakamoto and Yamamoto [6] in which they raised the question of global solutions to the equation

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$$ {}^{c}D^{\alpha}u = -u(1-u), \quad t > 0. \tag{1.4}$$

Their question has been solved positively by Hnaien, Kellil and Lassoued [3]. Our problem (1.1)–(1.3) is a natural extension of (1.4). We will prove the existence of globally bounded solutions as well as blowing-up solutions according to the condition imposed on the initial data. Let us mention that with the change of variable $v := 1-u$, (1.1) is transformed to the KPP-Fisher equation for both the homogeneous Neumann boundary condition or the case of a space variable in the Euclidean space. One may expect an analysis similar to the one for the KPP–Fisher equation modulo the influence of the time fractional Caputo derivative; however for the Dirichlet boundary condition, v has to satisfy $v_{|\partial\Omega} = 1$.

2. Preliminary results

As it is now known, the mild solution to problem (1.1)–(1.3) can be written, in the case of a bounded domain, into the form

$$u(t) = E_{\alpha}(-t^{\alpha}\mathcal{A})u_0 + \alpha\int_0^t s^{\alpha-1}E'_{\alpha}(-s^{\alpha}\mathcal{A})f(u(t-s))\,ds, \tag{2.1}$$

where $E_{\alpha}(z)$ is the Mittag–Leffler function

$$E_{\alpha}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)},$$

the linear operator $E_{\alpha}(-t^{\alpha}\mathcal{A})$ in (2.1), where $\mathcal{A}$ is the L^2 realization of the Laplacian $-\Delta$, is given by the standard operator calculus for self-adjoint operators, and we have set $f(u) = u(u-1)$.

We have the two important inequalities

$$\|E_{\alpha}(-t^{\alpha}\mathcal{A})u_0\|_{L^{\infty}} \leqslant \|u_0\|_{L^{\infty}}, \tag{2.2}$$

and

$$\|E'_{\alpha}(-t^{\alpha}\mathcal{A})u_0\|_{L^{\infty}} \leqslant \frac{1}{\alpha\Gamma(\alpha)}\|u_0\|_{L^{\infty}}. \tag{2.3}$$

In the case of $\mathbb{R}^N$, the integral form is

$$u(x,t) = \int_{\mathbb{R}^N} S_{\alpha,1}(x-y,t)\,dy + \int_0^t\int_{\mathbb{R}^N} G_{\alpha}(x-y,t-s)f(u(x,s))\,ds\,dx, \tag{2.4}$$

where

$$S_{\alpha,1}(x,t) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ipx}E_{\alpha,1}(-|p|^2t^{\alpha})\,dp$$

and

$$G_{\alpha}(x,t) = \frac{t^{\alpha-1}}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ipx}E_{\alpha,\alpha}(-|p|^2t^{\alpha})\,dp,$$

where

$$E_{\alpha,\alpha}(z) = \sum_{k=1}^{\infty}\frac{z^k}{\Gamma(k\alpha+\alpha)}.$$

• **Existence of local solutions.**

Theorem 2.1. *Assume that u_0 is continuous. Then problem (1.1)–(1.3) admits a local mild solution $u \in C([0,T_{max}[, C(\mathcal{D}))$ with the alternative:*

- *either $T_{max} = +\infty$;*
- *or $T_{max} < +\infty$ and in this case $\lim_{t\to T_{max}}\|u(t)\|_{L^{\infty}} = +\infty$.*

Proof. We give the proof only in the case of a bounded domain; the proof for the Euclidean space is similar. For $R \in (0,+\infty)$, we set

$$\mathcal{B} = \left\{u \in C([0,\tau], C(\mathcal{D})),\ \sup_{t\in[0,\tau]}\|u(t)-u_0\|_{\infty} \leqslant R\right\}, \quad \mathcal{D} = \Omega \text{ or } \mathbb{R}^N, \tag{2.5}$$

where τ will be specified later.