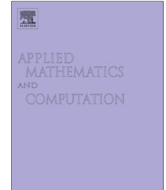




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On a time fractional reaction diffusion equation

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ABSTRACT

A reaction diffusion equation with a Caputo fractional derivative in time and with various boundary conditions is considered. Under some conditions on the initial data, we show that solutions may experience blow-up in a finite time. However, for realistic initial conditions, solutions are global in time. Moreover, the asymptotic behavior of bounded solutions will be analyzed.

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1. Introduction

In this paper, we consider the reaction diffusion system

$${}^c D^\alpha u - \Delta u = -u(1 - u), \quad x \in \Omega, \quad t > 0, \quad (1.1)$$

supplemented with:

– the homogeneous boundary condition

$$Bu = 0, \quad x \in \partial\Omega, \quad t > 0, \quad (1.2)$$

where $Bu = u|_{\partial\Omega}$ or $Bu = \frac{\partial u}{\partial \nu}|_{\partial\Omega}$, ν the outward normal to $\partial\Omega$ (if $\Omega = \mathbb{R}^N$, the condition $Bu = 0$ is omitted but we add $\lim_{|x| \rightarrow \infty} u(x, t) = 0$),

– and with the initial condition

$$u(x, 0) = u_0(x), \quad x \in \Omega; \quad (1.3)$$

the initial data $u_0(x)$ is a given positive and bounded function. Here $d > 0$ is the diffusion coefficient and ${}^c D^\alpha$ is the time fractional Caputo derivative of order $0 < \alpha < 1$ defined by

$${}^c D^\alpha u(x, t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{\partial u}{\partial s}(x, s)(t - s)^{-\alpha} ds$$

for a function u differentiable in the time variable [4,10].

Our paper is motivated by the recent one of Nakagawa, Sakamoto and Yamamoto [6] in which they raised the question of global solutions to the equation

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$${}^c D^\alpha u = -u(1-u), \quad t > 0. \quad (1.4)$$

Their question has been solved positively by Hnaien, Kellil and Lassoued [3]. Our problem (1.1)–(1.3) is a natural extension of (1.4). We will prove the existence of globally bounded solutions as well as blowing-up solutions according to the condition imposed on the initial data. Let us mention that with the change of variable $v := 1 - u$, (1.1) is transformed to the KPP-Fisher equation for both the homogeneous Neumann boundary condition or the case of a space variable in the Euclidean space. One may expect an analysis similar to the one for the KPP-Fisher equation modulo the influence of the time fractional Caputo derivative; however for the Dirichlet boundary condition, v has to satisfy $v|_{\partial\Omega} = 1$.

2. Preliminary results

As it is now known, the mild solution to problem (1.1)–(1.3) can be written, in the case of a bounded domain, into the form

$$u(t) = E_\alpha(-t^\alpha \mathcal{A})u_0 + \alpha \int_0^t s^{\alpha-1} E'_\alpha(-s^\alpha \mathcal{A})f(u(t-s)) ds, \quad (2.1)$$

where $E_\alpha(z)$ is the Mittag-Leffler function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)},$$

the linear operator $E_\alpha(-t^\alpha \mathcal{A})$ in (2.1), where \mathcal{A} is the L^2 realization of the Laplacian $-\Delta$, is given by the standard operator calculus for self-adjoint operators, and we have set $f(u) = u(u-1)$.

We have the two important inequalities

$$\|E_\alpha(-t^\alpha \mathcal{A})u_0\|_{L^\infty} \leq \|u_0\|_{L^\infty}, \quad (2.2)$$

and

$$\|E'_\alpha(-t^\alpha \mathcal{A})u_0\|_{L^\infty} \leq \frac{1}{\alpha \Gamma(\alpha)} \|u_0\|_{L^\infty}. \quad (2.3)$$

In the case of \mathbb{R}^N , the integral form is

$$u(x, t) = \int_{\mathbb{R}^N} S_{\alpha,1}(x-y, t) dy + \int_0^t \int_{\mathbb{R}^N} G_\alpha(x-y, t-s) f(u(x, s)) ds dx, \quad (2.4)$$

where

$$S_{\alpha,1}(x, t) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ipx} E_{\alpha,1}(-|p|^2 t^\alpha) dp$$

and

$$G_\alpha(x, t) = \frac{t^{\alpha-1}}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ipx} E_{\alpha,\alpha}(-|p|^2 t^\alpha) dp,$$

where

$$E_{\alpha,\alpha}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(k\alpha + \alpha)}.$$

• Existence of local solutions.

Theorem 2.1. Assume that u_0 is continuous. Then problem (1.1)–(1.3) admits a local mild solution $u \in C([0, T_{\max}], C(\mathcal{D}))$ with the alternative:

- either $T_{\max} = +\infty$;
- or $T_{\max} < +\infty$ and in this case $\lim_{t \rightarrow T_{\max}} \|u(t)\|_{L^\infty} = +\infty$.

Proof. We give the proof only in the case of a bounded domain; the proof for the Euclidean space is similar. For $R \in (0, +\infty)$, we set

$$\mathcal{B} = \left\{ u \in C([0, \tau], C(\mathcal{D})), \sup_{t \in [0, \tau]} \|u(t) - u_0\|_{L^\infty} \leq R \right\}, \quad \mathcal{D} = \Omega \text{ or } \mathbb{R}^N, \quad (2.5)$$

where τ will be specified later.

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