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Two fractional derivative inclusion problems via integral boundary condition

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ABSTRACT

The goal of the manuscript is to analyze the existence of solutions for the Caputo fractional differential inclusion ${}^cD^qx(t) \in F(t, x(t), {}^cD^{\beta}x(t))$ with the boundary value conditions $x(0) = 0$ and $x(1) + x'(1) = \int_0^{\eta} x(s)ds$, such that $0 < \eta < 1$, $1 < q \le 2$, $0 < \beta < 1$ and $q - \beta > 1$. Also, we investigate the existence of solutions for the Caputo fractional differential inclusion ${}^cD^qx(t) \in F(t, x(t))$ such that $x(0) = a\int_0^{\nu} x(s)ds$ and $x(1) = b\int_0^{\eta} x(s)ds$, where $0 < \nu, \eta < 1$, $1 < q \le 2$ and $a, b \in \mathbb{R}$.

1. Introduction

During the last decade the fractional differential equations were developed intensively (see for example, [1–16,26] and the references therein). On the other hand, a great attention was devoted to the fractional differential inclusions (see for example, [17,18,20,21,24,27] and the references therein). We recall that the Riemann–Liouville fractional integral of order $\alpha > 0$ of $f : (0, \infty) \to \mathbb{R}$ is $I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}f(s)ds$ if the right side is pointwise defined on $(0, \infty)$ (see [22,28,29]). The definition of the Caputo fractional derivative is ${}^cD^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds$, where $n = [\alpha] + 1$ (see [22,28,29]).

We consider (X, d) to be a metric space and denote by $P(X)$ and 2^X the class of all subsets and the class of all nonempty subsets of X, respectively. As a result, we denote the class of all closed, bounded and compact subsets of X by $P_{cl}(X), P_{bd}(X)$ and $P_{cp}(X)$, respectively. A mapping $Q : X \to 2^X$ is called a multifunction on X and $u \in X$ is called a fixed point of Q whenever $u \in Qu$ ([19]). Let us consider $J = [0, 1]$. A multifunction $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable whenever the function $t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable for all $y \in \mathbb{R}$ ([19]). By utilizing some fixed point results, the main aim of our work is to investigate the existence of solutions for the fractional differential inclusions presented in the abstract. For this purpose we consider the Hausdorff metric $H_d : 2^X \times 2^X \to [0, \infty)$ by $H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}$, where $d(A, b) = \inf_{a\in A} d(a; b)$. We recall that $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space ([19,23]).

Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ a map and $T : X \to 2^X$ a multifunction. We say that X has the condition (C_{α}) whenever for each sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such

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that $\alpha(x_{n_k}, x) \geqslant 1$ for all k. We claimed that T is α-admissible whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geqslant 1$, we have $\alpha(y,z) \geqslant 1$ for all $z \in Ty$ ([25]). Suppose that Ψ is the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$.

Lemma 1.1 [25]. *Let (X,d) be a complete metric space, $\alpha : X \times X \to [0,\infty)$ a map, $\psi \in \Psi$ a strictly increasing map and $T : X \to CB(X)$ a α-admissible multifunction such that $\alpha(x,y)H_d(Tx,Ty) \leqslant \psi(d(x,y))$ for all $x,y \in X$ and there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0,x_1) \geqslant 1$. If X has the condition (C_α), then T has a fixed point.*

Lemma 1.2 [4]. *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to P_{cp,cv}(C)$ is an upper semi-continuous compact map, where $P_{cp,cv}(C)$ denotes the family of nonempty, compact convex subsets of C. Then either F has a fixed point in $\overline{U}$ or there exist $u \in \partial U$ and $\lambda \in (0,1)$ such that $u \in \lambda F(u)$.*

2. Main results

In the first step we analyze the fractional derivative inclusion

$$^cD^q x(t) \in F(t, x(t), {}^cD^\beta x(t)) \qquad (*)$$

such that $x(1) + x'(1) = \int_0^\eta x(s)ds$ and $x(0) = 0$, where $t \in J, \beta, \eta \in (0,1), q \in (1,2]$ with $q - \beta > 1$, $^cD^q$ is the Caputo differentiation and $F : J \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ denotes a compact valued multifunction.

Lemma 2.1. *Let $v \in C(J,\mathbb{R})$. Then, the unique solution of the fractional integral boundary value problem $^cD^q x(t) = v(t)$ via the boundary value problems $x(1) + x'(1) = \int_0^\eta x(s)ds$ and $x(0) = 0$, where $\beta, \eta \in (0,1), q \in (1,2]$ with $q - \beta > 1$, is given by*

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s)ds + \frac{2t}{(4-\eta^2)\Gamma(q)} \int_0^\eta \int_0^s (s-m)^{q-1} v(m)dmds - \frac{2t}{(4-\eta^2)\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds \\ &\quad - \frac{2t}{(4-\eta^2)\Gamma(q-1)} \int_0^1 (1-s)^{q-2} v(s)ds \\ &= P_v(t) + \int_0^1 G(t,s) v(s)ds, \end{aligned}$$

where $P_v(t) = \frac{2t}{(4-\eta^2)\Gamma(q)} \int_0^\eta \int_0^s (s-m)^{q-1} v(m)dmds$ and

$$G(t,s) = \begin{cases} \frac{(4-\eta^2)(t-s)^{(q-1)} - 2t(1-s)^{q-1}}{(4-\eta^2)\Gamma(q)} - \frac{2t(1-s)^{q-2}}{(4-\eta^2)\Gamma(q-1)} & 0 < s < t < 1, \\ \frac{-2t(1-s)^{q-1}}{(4-\eta^2)\Gamma(q)} - \frac{2t(1-s)^{q-2}}{(4-\eta^2)\Gamma(q-1)} & 0 < t < s < 1. \end{cases}$$

Proof. We recall the general solution of the equation $^cD^q x(t) = v(t)$ as

$$x(t) = I^q v(t) - c_0 - c_1 t = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} v(s)ds - c_0 - c_1 t,$$

where $c_0, c_1 \in \mathbb{R}$ denote arbitrary constants. Thus,

$$x'(t) = I^{(q-1)} v(t) - c_1 = \frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} v(s)ds - c_1.$$

By utilizing the boundary conditions, we conclude $c_0 = 0$ and

$$c_1 = -\frac{2}{(4-\eta^2)\Gamma(q)} \int_0^\eta \int_0^s (s-m)^{q-1} v(m)dmds + \frac{2}{(4-\eta^2)\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds + \frac{2}{(4-\eta^2)\Gamma(q-1)} \int_0^1 (1-s)^{q-2} v(s)ds.$$

Hence,

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s)ds + \frac{2t}{(4-\eta^2)\Gamma(q)} \int_0^\eta \int_0^s (s-m)^{q-1} v(m)dmds - \frac{2t}{(4-\eta^2)\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds \\ &\quad - \frac{2t}{(4-\eta^2)\Gamma(q-1)} \int_0^1 (1-s)^{q-2} v(s)ds = P_v(t) + \int_0^1 G(t,s) v(s)ds. \end{aligned}$$

This completes our proof. □