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The spectral analysis for a singular fractional differential equation with a signed measure

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A R T I C L E I N F O

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A B S T R A C T

In this paper, by using the spectral analysis of the relevant linear operator and Gelfand's formula, we obtain some properties of the first eigenvalue of a fractional differential equation. Based on these properties, the fixed point index of the nonlinear operator is calculated explicitly and some sufficient conditions for the existence of positive solutions are established.

1. Introduction

In linear elastic fracture mechanics, the stress near the crack tip exhibits a singularity of $r^{-0.5}$ [1], where r is the distance measured from the crack tip. This classical singularity also exists in many physical phenomena and biological processes, such as gas dynamics, Newtonian fluid mechanics, nuclear physics, engineering sciences and infectious disease. So the study of singularity of differential equations has attracted much attention in recent years, and for details the reader is referred to [2–6,8–12] and the references cited therein.

In this paper, we study a singular fractional differential equation with signed measure

$$\begin{cases} (-\mathcal{D}_t^{\alpha}x)(t) = f(t, x(t), \quad \mathcal{D}_t^{\beta}x(t)), t \in (0,1), \\ \mathcal{D}_t^{\beta}x(0) = 0, \quad \mathcal{D}_t^{\beta}x(1) = \int_0^1 \mathcal{D}_t^{\beta}x(s)dA(s), \end{cases} \tag{1.1}$$

where $\mathcal{D}_t^{\alpha}$, $\mathcal{D}_t^{\beta}$ are the standard Riemann–Liouville derivatives, $\int_0^1 x(s)dA(s)$ is denoted by a Riemann–Stieltjes integral and $0 < \beta \le 1 < \alpha \le 2$, $\alpha - \beta > 1$, A is a function of bounded variation and dA can be a signed measure, the nonlinearity $f(t, x, y)$ may be singular at both $t = 0$, 1 and $x = y = 0$.

Here we also review some details on the nonlocal boundary condition given by a Riemann–Stieltjes integral with a signed measure. The linear functional $\int_0^1 \mathcal{D}_t^{\beta}x(s)dA(s)$ in (1.1) is given by a Riemann–Stieltjes integral with A being a suitable function of bounded variation. Thus A can include both sums and integrals, which implies that the nonlocal boundary condition of Riemann–Stieltjes integral type is a more general case than the multi-point boundary condition and the integral boundary condition ($A(s) = s$ or $dA(s) = h(s)ds$). Moreover, dA can be a signed measure on a subset of $[0,1]$ (see [7]).

During the last decade, fractional equations have been discussed extensively as valuable tools in the modeling of many phenomena in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology,

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economics, control theory, signal and image processing, aerodynamics, viscoelasticity, electromagnetics, and so on (see [17–24]). Recently, many manuscripts on fractional differential equations were published (see, e.g., [2,3,5,14,25–33] and the references therein), and many differential equations have been discussed under nonlocal boundary conditions for a wide range of applications. For example, integral boundary conditions arise in thermal conduction problems [34]; semiconductor problems [35] and hydrodynamic problems [36]; coupled boundary conditions appear in the study of reaction–diffusion equations and Sturm–Liouville problems [37,38] and have applications in many fields of sciences and engineering [39–41] and mathematical biology [42].

Due to the profound background of the above phenomena, this work mainly focus on handling the singularity, in particular, the singularity of f at $x = y = 0$, and the nonlocal conditions for the problem (1.1). By using the spectral analysis of the relevant linear operator and Gelfand's formula, we obtain some properties of the first eigenvalue corresponding to the relevant linear operator. Based on these properties, the fixed point index of the nonlinear operator is calculated explicitly and some sufficient conditions for the existence of positive solutions are established.

2. Preliminaries and lemmas

In this paper, we shall work in the Banach space $E = C[0,1]$ with the norm $\|y\| = \max_{t\in[0,1]}|y(t)|$. Let $P = \{y \in E : y(t) \geqslant 0,\ t \in [0,1]\}$ be a cone in E and construct a subcone of P as follows

$$K = \left\{ y \in P : y(t) \geqslant \frac{t^{\alpha-\beta-1}(1-t)}{\alpha-\beta}\|y\|,\ t \in [0,1] \right\}.$$

For any $r > 0$, let $K_r = \{y \in K : \|y\| < r\}$, $\partial K_r = \{y \in K : \|y\| = r\}$ and $\overline{K}_r = \{y \in K : \|y\| \leqslant r\}$.

Now we begin our work based on the theory of fractional calculus, and for details on definitions and semigroup properties of Riemann–Liouville fractional calculus, we refer the reader to [17–19].

Let $x(t) = I^{\beta}y(t),\ y(t) \in C[0,1]$, then it follows from the definition of Riemann–Liouville fractional derivative that

$$\begin{aligned} &\mathcal{D}_t^{\alpha}x(t) = \frac{d^n}{dt^n}I^{n-\alpha}x(t) = \frac{d^n}{dt^n}I^{n-\alpha}I^{\beta}y(t) = \frac{d^n}{dt^n}I^{n-\alpha+\beta}y(t) = \mathcal{D}_t^{\alpha-\beta}y(t),\\ &\mathcal{D}_t^{\beta}x(t) = \mathcal{D}_t^{\beta}I^{\beta}y(t) = y(t). \end{aligned} \tag{2.1}$$

Thus by applying (2.1), the BVP (1.1) reduces to the following modified boundary value problem

$$\begin{cases} (-\mathcal{D}_t^{\alpha-\beta}y)(t) = f(t, I^{\beta}y(t), y(t)),\\ y(0) = 0,\ y(1) = \int_0^1 y(s)dA(s). \end{cases} \tag{2.2}$$

Similarly, using (2.1) again, the BVP (2.2) is also transformed to the form (1.1). Thus the BVP (2.2) is indeed equivalent to the BVP (1.1).

Let

$$G(t,s) = \frac{1}{\Gamma(\alpha-\beta)} \begin{cases} [t(1-s)]^{\alpha-\beta-1}, & 0 \leqslant t \leqslant s \leqslant 1,\\ [t(1-s)]^{\alpha-\beta-1} - (t-s)^{\alpha-\beta-1}, & 0 \leqslant s \leqslant t \leqslant 1. \end{cases} \tag{2.3}$$

Lemma 2.1 (see [12]). *Given $h \in L^1(0,1)$, then the problem*

$$\begin{cases} -\mathcal{D}_t^{\alpha-\beta}y(t) = h(t), \quad 0 < t < 1,\\ y(0) = y(1) = 0, \end{cases} \tag{2.4}$$

has the unique solution

$$y(t) = \int_0^1 G(t,s)h(s)ds.$$

By Lemma 2.1, the unique solution of the problem

$$\begin{cases} -\mathcal{D}_t^{\alpha-\beta}y(t) = 0, \quad 0 < t < 1,\\ y(0) = 0, y(1) = 1, \end{cases} \tag{2.5}$$

is $t^{\alpha-\beta-1}$. Let

$$\mathcal{A} = \int_0^1 t^{\alpha-\beta-1}dA(t), \quad \mathcal{G}_A(s) = \int_0^1 G(t,s)dA(t).$$