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Some new results for multi-valued fractional evolution equations

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ABSTRACT

Given a closed linear operator A in a Banach space X and a multi-valued function with convex, closed values $F : [0, b] \times C([-\tau, 0]; X) \to 2^X$, we consider the Cauchy problem

$$\begin{cases} {}_cD_t^\alpha u(t) \in Au(t) + F(t, u_t), & t \in [0, b], \\ u(t) = \varphi(t), & t \in [-\tau, 0], \end{cases}$$

where $\tau \geqslant 0$, ${}_cD_t^\alpha$, $0 < \alpha < 1$, represents the regularized Caputo fractional derivative of order α. We concentrate on the case when the semigroup generated by A is noncompact and obtain nonemptyness of the solution set if, in particular, X is reflexive and F is weakly upper semicontinuous with respect to the second variable. Furthermore, in this situation topological properties of the set of all solutions are considered.

1. Introduction and motivations

Since the work of Mandelbrot [17], fractional calculus has been used successfully to study many complex systems in various fields of science and engineering. Overviews on applications of fractional calculus may be found in the books [11,21]. Fractional differential equations become very popular for describing various phenomena, for instance in viscoelasticity [2], anomalous diffusion [18] notably in chaotic systems [31], or in phase transitions [10]. At present, much interest has developed regarding the class of equations in infinite dimensional spaces, we refer the reader to [7,13–16,25–27,32,33], among others.

In the present paper we shall study the multi-valued perturbations of fractional evolution equations. More precisely, the problem we deal with is

$$\begin{cases} {}_cD_t^\alpha u(t) \in Au(t) + F(t, u_t), & t \in [0, b], \\ u(t) = \varphi(t), & t \in [-\tau, 0] \end{cases} \tag{1.1}$$

in a real Banach space $(X, \|\cdot\|)$, where $\tau \geqslant 0$, $A : D(A) \subset X \to X$ (possibly unbounded) is a linear closed operator, ${}_cD_t^\alpha$, $0 < \alpha < 1$, represents the regularized Caputo fractional derivative of order α, $u_t \in C([-\tau, 0]; X)$ is defined by $u_t(s) = u(t + s)(s \in [-\tau, 0])$, $\varphi \in C([-\tau, 0]; X)$, and $F : [0, b] \times C([-\tau, 0]; X) \to 2^X$ is a multi-valued function with convex closed values. We consider perturbations F being weakly upper semicontinuous with respect to the second variable.

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We mention that the setting determines the necessity to use the regularized fractional derivative (see (2.1) below). If, for example, one considers instead of this the Riemann–Liouville fractional derivative, but without subtracting $t^{-\alpha}u(0)$, then the appropriate initial data will be the limit value, as $t \to 0$, of the fractional integral of a solution of the order $1-\alpha$, not the limit value of the solution itself.

It is known that Cauchy problems of type (1.1) are abstract models of many problems involving retarded differential equations and inclusions; recently, there has been a great literature devoted to studying the solvability, see, e.g., [20,24–28,32] and the references therein. We mention that in the case when F is multi-valued, much of the previous research was done provided that, in particular, F with compact values, is upper semicontinuous with respect to the solution variable. As indicated in [5] (see also [23]), this condition is restrictive to some extent and not satisfied usually in practical applications. To make things more applicable, an appropriate condition for F is that as we mentioned previously.

In the study of various differential equations and inclusions, one of the most important problems is the topological structure of the solution sets (such as nonemptyness, compactness, R_δ-structure, etc.), which as an interesting qualitative property plays a key role in discussing the solvability, approximate controllability and periodicity, etc for deterministic problems; some of the more recent research are, e.g., [1,8,28] and one can find further references therein. In particular, in our previous work [5], we studied the topological structure of the solution sets to the Cauchy problem of nonlinear delay evolution inclusion

$$\begin{cases} u'(t) \in Au(t) + F(t, u(t), u_t), \\ u(t) = \phi(t), \quad t \in [-\tau, 0] \end{cases}$$

both on compact intervals and non-compact intervals, where A is an m-dissipative operator (possible multi-valued and/or nonlinear) and F is a multi-valued function with convex, closed values. Moreover, we used the information of the structure to show the existence of global C^0-solutions for the evolution inclusion subject to nonlocal initial condition. As can be seen, a crucial assumption in [5] is that the nonlinear semigroup generated by A is compact.

In this paper, we shall study topological properties of the solution set for Cauchy problem (1.1), concentrating on the case when the semigroup generated by A is noncompact. More precisely, the main purpose is to establish nonemptyness, compactness of the solution set and upper semicontinuity of the solution operator, as the semigroup generated by A is only continuous for $t > 0$ in the uniform operator topology.

Here u is called a solution of Cauchy problem (1.1) if $u \in C([-\tau, b]; X)$ is the mild solution of fractional evolution equation subject to time delay

$$\begin{cases} {}_cD_t^\alpha u(t) = Au(t) + f(t), \quad t \in [0, b], \\ u(t) = \varphi(t), \quad t \in [-\tau, 0] \end{cases} \tag{1.2}$$

with some

$$f \in Sel(u) := \{f \in L^p(0, b; X); f(t) \in F(t, u_t) \text{ for a.e. } t \in [0, b]\},$$

where $p > 1$ and $p\alpha > 1$. As can be seen from Definition 2.3 below, the concept of the solution involves a singular integral equation.

Let Wf denote the unique mild solution of (1.2) corresponding to f (for fixed φ). For $U \subset C([-\tau, 0]; X)$ and $t \in [-\tau, 0]$ fixed, we denote $U(t) = \{w(t) : w \in U\}$. Let us give a short description for the problems we encounter. Loss of compactness of the semigroup generated by A enables us to have to impose the following condition on F, namely

(1) there exists $\eta \in C([0, b]; \mathbb{R}^+)$ such that $\chi(F(t, \Omega)) \leqslant \eta(t) \sup_{s \in [-\tau, 0]} \chi(\Omega(s))$ for a.e. $t \in [0, b]$ and each bounded subset $\Omega \subset C([-\tau, 0]; X)$,

where $\chi(\cdot)$ denotes the Hausdorff-measure of noncompactness in X; see [6, Section 9.2]. Then, to overcome the difficulty caused by singularity of solutions, one has to impose the special assumptions on F to get a L^p-integrable selection of F, which yields that the Nemytskii operator

$$Sel : C([-\tau, b]; X) \to 2^{L^p(0, b; X)}$$

is a multi-valued map. Moreover, when the semigroup generated by A is noncompact, one technical difficulty that arises is to find a compact convex subset of $C([-\tau, b]; X)$ which is invariant under the operator defined by

$$G := W \circ Sel.$$

Let us notice that in the previous work such as [5], when the semigroup is compact, the compactness of the solution sets become a direct consequence of the proof of the existence. In the present situation, however, we'll have to deal with this problem, since it is chained down to loss of compactness of the semigroup.

As the reader will see, our results are non comparable variants of those given in the aforementioned papers.

Our work is arranged as follows. Section 2 is devoted to introducing some notations, establishing some conventions and describing some results which are essential tools in the subsequent sections. In Section 3, we establish the existence of solutions of Cauchy problem (1.1) under various situations. Compactness of the solution set and property of the corresponding solution operator are then considered. In Section 4, we present an example to illustrate the effectiveness of our main theoretical results.