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Existence of mild solution for evolution equation with Hilfer fractional derivative

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ABSTRACT

The paper is concerned with existence of mild solution of evolution equation with Hilfer fractional derivative which generalized the famous Riemann–Liouville fractional derivative. By noncompact measure method, we obtain some sufficient conditions to ensure the existence of mild solution. Our results are new and more general to known results.

1. Introduction

Nowadays, fractional calculus receives increasing attention in the scientific community, with a growing number of applications in physics, electrochemistry, biophysics, viscoelasticity, biomedicine, control theory, signal processing, etc(see [22,29] and the references therein). Fractional differential equations also have been proved to be useful tools in the modeling of many phenomena in various fields of science and engineering. There has been a significant development in fractional differential equations in recent years, see the monographs of Kilbas et al. [16], Miller and Ross [21], Podlubny [23], Lakshmikantham et al. [17], Zhou [32], the papers [1,4–7,9,11,18,19,27,28] and the references therein.

A strong motivation for investigating fractional evolution equations comes from physics. Fractional diffusion equations are abstract partial differential equations that involve fractional derivative in space and time. For example, EI-Sayed [10] discussed fractional order diffusion-wave equation. Eidelman and Kochubei [11] investigated the Cauchy problem for fractional diffusion equation. As stated in [11], fractional diffusion equations describe anomalous diffusion on fractals. Physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials. This class of equations can provide a nice instrument for the description of memory and hereditary properties of various materials and processes.

Some recent papers investigated the problem of the existence of mild solution for abstract differential equations with fractional derivative [2,8,15,25]. Since the mild solution definition in integer order abstract differential equations obtained by variation of constant formulas can not be generalized directly to fractional order abstract differential equations, Zhou and Jiao [30] gave a suit concept on mild solutions by applying laplace transform and probability density functions for evolution equation with Caputo fractional derivative. Using the same method, Zhou et al. [31] gave a suit concept on mild solutions for evolution equation with Riemann–Liouville fractional derivative. By using sectorial operator, Su et al. [25] gave a definition of mild solution for fractional differential equation with order $1 < \alpha < 2$ and investigated it's existence. Agarwal et al. [2] studied the existence and dimension of the set for mild solutions of semilinear fractional differential equations inclusions. Wang [26] researched the abstract fractional Cauchy problem with almost sectorial operators. On the other hand, Hilfer [14]

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proposed a generalized Riemann–Liouville fractional derivative, for short, Hilfer fractional derivative, which includes Riemann–Liouville fractional derivative and Caputo fractional derivative. This operator appeared in the theoretical simulation of dielectric relaxation in glass forming materials. In [12], Furati et al. considered an initial value problem for a class of nonlinear fractional differential equations involving Hilfer fractional derivative. In [24], the solution of a fractional diffusion equation with a Hilfer time fractional derivative was obtained in terms of Mittag–Leffler functions and Fox's H-function. To the best of our knowledge, there has no results about the evolution equations with Hilfer fractional derivative.

Inspired by the above discussion, in this paper, we will investigate a class of evolution equation with Hilfer fractional derivative. By Laplace transform and density function, we firstly give the mild solution definition. Then we obtain some sufficient conditions ensuring the existence of mild solution by using noncompact measure method and Ascoli–Arzela Theorem. Because Hilfer fractional derivative is more general than Riemann–Liouville fractional derivative. So, the results we obtained are also more general than known results.

In this paper, we consider the following fractional order evolution equation:

$$\begin{cases} D_{0+}^{\nu,\mu}x(t) = Ax(t) + f(t,x(t)), & t \in J' = (0,b], \\ I_{0+}^{(1-\nu)(1-\mu)}x(0) = x_0, \end{cases} \tag{1.1}$$

where $D_{0+}^{\nu,\mu}$ is the Hilfer fractional derivative which will be given in next section, $0 \leqslant \nu \leqslant 1$, $0 < \mu < 1$, the state $x(\cdot)$ takes value in a Banach space X with norm $|\cdot|$, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators (i.e. C_0 semigroup) $\{Q(t)\}_{t\geqslant 0}$ in Banach space X, $f : J \times X \to X$ is given functions satisfying some assumptions, $x_0 \in X$.

The rest of this paper is organized as follows. In Section 2, some notations and preparation are given. A suitable concept on mild solution for our problem is introduced. In Section 3, some sufficient conditions are obtained to ensure the existence of mild solution. Some conclusions are given in Section 4.

2. Preliminaries

In this section, we will firstly introduce fractional integral and derivative, some notations and then give the definition of a mild solution of system (1.1). Finally, we will give some assumptions and lemmas which are useful in next section.

Throughout this paper, $\mathbb{R}$ represents the set of real numbers, and $\mathbb{R}_+ = [0,\infty)$. Let $J = [0,b]$ and $J' = (0,b]$, by $C(J,X)$ and $C(J',X)$ we denote the spaces of all continuous functions from J to X and J' to X, respectively.

Define

$$Y = \{x \in C(J',X) : \lim_{t\to 0} t^{(1-\nu)(1-\mu)}x(t) \text{ exists and infinite}\},$$

with the norm $\|\cdot\|_Y$ defined by

$$\|x\|_Y = \sup_{t\in J'} |t^{(1-\nu)(1-\mu)}x(t)|.$$

Obviously, Y is a Banach space. We also note that:

(i) When $\nu = 1$, then $Y = C(J,X)$ and $\|\cdot\|_Y = \|\cdot\|$
(ii) Let $x(t) = t^{(\nu-1)(1-\mu)}y(t)$ for $t \in J'$, $x \in Y$ if and only if $y \in C(J,X)$, and $\|x\|_Y = \|y\|$.

Let $B_r(J) = \{y \in C(J,X) | \|y\| \leqslant r\}$ and $B_r^Y(J') = \{x \in Y | \|x\|_Y \leqslant r\}$, then B_r and B_r^Y are two bounded closed and convex subsets of $C(J,X)$ and Y, respectively.

Definition 2.1 (*see [23]*). The fractional integral of order p with the lower limit a for a function $f : [a,\infty) \to \mathbb{R}$ is defined as

$$I_{a+}^{p}f(t) = \frac{1}{\Gamma(p)}\int_a^t \frac{f(s)}{(t-s)^{1-p}}ds, \; t > a, \; p > 0,$$

provided the right side is point-wise defined on $[a,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 (*see [23]*). The Riemann–Liouville derivative of order $p > 0$ for a function $f : [a,\infty) \to \mathbb{R}$ is defined as

$$D_{a+}^{p}f(t) = \frac{1}{\Gamma(n-p)}\frac{d^n}{dt^n}\int_a^t \frac{f(s)}{(t-s)^{p+1-n}}ds, \; t > a, \; n-1 < p < n.$$

Definition 2.3 (*see [23]*). The Caputo derivative of order $p > 0$ for a function $f : [a,\infty) \to \mathbb{R}$ is defined as

$${}^{C}D_{a+}^{p}f(t) = D_{a+}^{p}[f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)], \; t > a, \; n-1 < p < n.$$