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Heat conduction modeling by using fractional-order derivatives ☆

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ABSTRACT

The article deals with the heat conduction modeling. A brief historical overview of the authors who have dealt with the heat conduction and overview of solving methods is listed in the introduction of article. In the next section a mathematical model of one-dimensional heat conduction with using derivatives of integer- and fractional-order is described. The methods of solving models of heat conduction are described, namely analytical and numerical methods. In the case of numerical methods regards the finite difference method by using Grünwald–Letnikov definition for the fractional time derivative. Implementation of these individual methods was realized in MATLAB. The two libraries of *m*-functions for the heat conduction model have been created, namely Heat Conduction Toolbox and Fractional Heat Conduction Toolbox. At the conclusion of the article the simulations examples with using toolboxes are listed.

1. Introduction

Heat conduction process, described by partial differential equation, was first formulated by Jean Baptiste Joseph Fourier (1768–1830). In 1807 he wrote an article "Partial differential equation for heat conduction in solids". The issue of heat conduction was addressed by other scientists as well, such as Fick, Maxwell, Einstein, Richards, Taylor [1].

The various analytical and numerical methods are used to the solution the Fourier heat conduction equation [2,3]. In the case of heat conduction in materials with non-standard structure, such as polymers, granular and porous materials, composite materials and so on, a standard description is insufficient and required the creation of more adequate models with using derivatives of fractional-order [4–8]. The causes are mainly memory systems and ongoing processes [9–13], roughness or porosity of the material [14–16] and also fractality and chaotic behavior of systems [17–26].

The issue of research and development methods and tools for processes modeling with using fractional-order derivatives is very actual, since it means a qualitatively new level of modeling. Important authors of the first articles were Fourier, Abel, Leibniz, Grünwald and Letnikov. Mathematicians like Liouville (1809–1882) [27,28] and Riemann (1826–1866) [29] made major contributions to the theory of fractional calculus.

Nowadays there are a number of analytical [30–37] and numerical solutions of fractional heat conduction equation. In the case of numerical methods are developed different methods based on the random walk models [38–41], the finite difference

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method (FDM) [42–44], the finite element method [45–48], numerical quadrature [49–51], the method of Adomian decomposition [52,53], Monte Carlo simulation [54,55], matrix approach [4,5,56] or the matrix transform method [57,58]. The finite difference method is an extended method where are used an explicit [42,59,60], an implicit [43,61–63], and a Crank–Nicolson scheme [44,64]. For the Crank–Nicolson scheme, the literature describes the use of Grünwald–Letnikov definition only for a spatial derivative [62,65–67].

The work presented in this article is mainly aimed at the implementation of FDM for the fractional heat conduction equation in MATLAB and in the case of Crank–Nicolson scheme brings the use of Grünwald–Letnikov definition for the time derivative. The article does not address the questions of the stability and convergency because they are described in detail in many works, e.g. [60,63,68].

2. Heat conduction models

A heat conduction is a molecular transfer of thermal energy in solids, liquids and gases due to the temperature difference. The process of the heat conduction takes place between the particles of the substance when they directly touch each-other and have different temperature. Existing models of heat conduction processes are divided according to various criterions. We consider a division into two groups to models with using derivatives of integer and fractional order.

2.1. Models with using derivatives of integer order

Models with using derivatives of integer order are the non-stationary and stationary models. Non-stationary models are described by Fourier heat conduction equation, where the temperature T (K) is a function of spatial coordinate x (m) and time τ (s). In the case of one-dimensional heat conduction it has the following form

$$\begin{aligned} &\frac{\partial T(x,\tau)}{\partial \tau} = \left(\sqrt{a}\right)^2 \frac{\partial^2 T(x,\tau)}{\partial x^2} \quad \text{for} \quad 0 < x < L \quad \text{and} \quad \tau > 0, \\ &T(0,\tau) = T_1, \quad T(L,\tau) = T_2 \quad \text{for} \quad \tau > 0, \\ &T(x,0) = f(x) \quad \text{for} \quad 0 \leqslant x \leqslant L, \end{aligned} \tag{1}$$

where $a = \lambda/(\rho \cdot c_p)$ is thermal diffusivity ($\text{m}^2\ \text{s}^{-1}$), ρ is density (kg m^{-3}), c_p is specific heat capacity ($\text{J kg}^{-1}\ \text{K}^{-1}$) and λ is thermal conductivity ($\text{W m}^{-1}\ \text{K}^{-1}$).

2.2. Models with using derivatives of fractional order

A more general formulation of the task for modeling not only one-dimensional heat conduction is based on the model in which on the left-hand side of the Eq. (1) instead of the first derivative with respect to time, the derivative of order α occurs, i.e. we can find it in the form

$$\begin{aligned} &\frac{\partial^\alpha T(x,\tau)}{\partial \tau^\alpha} = (b)^2 \frac{\partial^2 T(x,\tau)}{\partial x^2} \quad \text{for} \quad 0 < x < L \quad \text{and} \quad \tau > 0, \\ &T(0,\tau) = T_1, \quad T(L,\tau) = T_2 \quad \text{for} \quad \tau > 0, \\ &T(x,0) = f(x) \quad \text{for} \quad 0 \leqslant x \leqslant L, \end{aligned} \tag{2}$$

where b represents a constant coefficient with the unit $\text{m} \cdot \text{s}^{-\alpha/2}$.

3. Methods of solution

Methods used to solve models (1) and (2) are divided into *analytical methods* and *numerical methods*.

3.1. Analytical methods

Analytical methods can be used for solving problems in a bounded, semi-bounded or unbounded interval. Analytical solution of heat conduction model (1) for a bounded interval $\langle 0, L\rangle$ is given by the following function, which corresponds to the sum of the product of trigonometric and exponential functions

$$T(x,\tau) = \sum_{k=1}^{\infty}\left[c_k\, e^{-(n\sqrt{a})^2\tau} \sin nx\right] + \frac{1}{L}(T_2 - T_1)x + T_1, \tag{3}$$

where

$$c_k = \frac{2}{L}\int_0^L \left[f(\xi) - \frac{1}{L}(T_2 - T_1)\xi - T_1\right] \sin\frac{k\pi\xi}{L}\, d\xi. \tag{4}$$