Applied Mathematics and Computation 257 (2015) 381–397

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients

Zhiyuan Li, Yikan Liu *, Masahiro Yamamoto

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

ARTICLE INFO

Keywords:
Initial-boundary value problem
Time-fractional diffusion equation
Multinomial Mittag–Leffler function
Well-posedness
Long-time asymptotic behavior
Laplace transform

ABSTRACT

In this paper, we investigate the well-posedness and the long-time asymptotic behavior for initial-boundary value problems for multi-term time-fractional diffusion equations. The governing equation under consideration includes a linear combination of Caputo derivatives in time with decreasing orders in $(0,1)$ and positive constant coefficients. By exploiting several important properties of multinomial Mittag–Leffler functions, various estimates follow from the explicit solutions in form of these special functions. Then we prove the uniqueness and continuous dependency on initial values and source terms, from which we further verify the Lipschitz continuous dependency of solutions with respect to coefficients and orders of fractional derivatives. Finally, by a Laplace transform argument, it turns out that the decay rate of the solution as $t \to \infty$ is given by the minimum order of the time-fractional derivatives.

1. Introduction

Let Ω be an open bounded domain in $\mathbb{R}^d$ with a smooth boundary (for example, of C^∞ class) and $T > 0$ be fixed arbitrarily. For a fixed positive integer m, let α_j and q_j $(j = 1, \ldots, m)$ be positive constants such that $1 > \alpha_1 > \cdots > \alpha_m > 0$. Consider the following initial-boundary value problem for the multi-term time-fractional diffusion equation

$$\begin{cases} \sum_{j=1}^{m} q_j \partial_t^{\alpha_j} u(x,t) = Lu(x,t) + F(x,t), & x \in \Omega,\ 0 < t \leqslant T, & (1.1)\\ u(x,t) = 0, & x \in \partial\Omega,\ 0 < t \leqslant T, & (1.2)\\ u(x,0) = a(x), & x \in \Omega, & (1.3) \end{cases}$$

where L is a symmetric uniformly elliptic operator with the homogeneous Dirichlet boundary condition, and we can assume $q_1 = 1$ without lose of generality. The regularities of the initial value a and the source term F will be specified later. Here $\partial_t^{\alpha_j}$ denotes the Caputo derivative defined by

$$\partial_t^{\alpha_j} f(t) := \frac{1}{\Gamma(1-\alpha_j)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha_j}} \, ds,$$

* Corresponding author.
E-mail addresses: zyli@ms.u-tokyo.ac.jp (Z. Li), ykliu@ms.u-tokyo.ac.jp (Y. Liu), myama@ms.u-tokyo.ac.jp (M. Yamamoto).

http://dx.doi.org/10.1016/j.amc.2014.11.073
0096-3003/

where $\Gamma(\cdot)$ is a usual Gamma function. For various properties of the Caputo derivative, we refer to Diethelm [6], Kilbas et al. [14], Podlubny [26] and Zhou [31]. See also [9,30] for further contents on fractional calculus. We abbreviate $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_m)$ and $\boldsymbol{q} := (q_1, \ldots, q_m)$ for later convenience.

In the case of $m = 1$, Eq. (1.1) is reduced to its single-term counterpart

$$\partial_t^\alpha u = Lu + F \quad \text{in} \quad \Omega \times (0, T], \; \alpha \in (0, 1). \tag{1.4}$$

The above formulation has been studied extensively from different aspects due to its vast capability of modeling the anomalous diffusion phenomena in highly heterogeneous aquifer and complex viscoelastic material (see [1,7,10,24] and the references therein). Indeed, although the single-term time-fractional diffusion equation inherits certain properties from the classical diffusion equation (i.e., $\alpha = 1$), it differs considerably from the traditional one especially in the senses of its limited smoothing effect in space and slow decay in time. In Luchko [18], a maximum principle of the initial-boundary value problem for (1.4) was established, and the uniqueness of a classical solution was proved. Luchko [19] represented the generalized solution to (1.4) with $F = 0$ by means of the Mittag–Leffler function and gave the unique existence result. Sakamoto and Yamamoto [28] carried out a comprehensive investigation including the well-posedness of the initial-boundary value problem for (1.4) as well as the long-time asymptotic behavior of the solution. It turns out that the spatial regularity of the solution is only moderately improved from that of the initial value, and the solution decays with order $t^{-\alpha}$ as $t \to \infty$. Recently, the Lipschitz stability of the solution to (1.4) with respect to α and the diffusion coefficient was proved as a byproduct of an inverse coefficient problem in Li et al. [15]. For other discussions concerning Eq. (1.4), see e.g., Gorenflo et al.[8], Luchko [17] and Prüss [27]. Regarding numerical treatments, we refer to Liu et al. [16] and Meerschaert and Tadjeran [22] for the finite difference method and Jin et al. [13] for the finite element method.

As a natural extension, Eq. (1.1) is expected to improve the modeling accuracy in depicting the anomalous diffusion due to its potential feasibility. However, to the authors' best knowledge, published works on this extension are quite limited in spite of rich literatures on its single-term version. Luchko [20] developed the maximum principle for problem (1.1)–(1.3) and constructed a generalized solution when $F = 0$ by means of the multinomial Mittag–Leffler functions. Jiang et al. [11] considered fractional derivatives in both time and space and derived analytical solutions. As for the asymptotic behavior, for $m = 2$ it reveals in Mainardi et al. [23] that the dominated decay rate of the solution is related to the minimum order of time fractional derivative. On the other hand, Beckers and Yamamoto [4] investigated (1.1)–(1.3) in a slightly more general formulation and obtained a weaker regularity result than that in [28]. Very recently, Jin et al. [12] developed semidiscrete and fully discrete Galerkin finite element methods for (1.1)–(1.3).

In this paper, we are concerned with the well-posedness and the long-time asymptotic behavior of the solution to the initial-boundary value problem (1.1)–(1.3), and we attempt to establish results parallel to that for the single-term case. On the basis of the explicit representation of the solution, by exploiting several properties of the multinomial Mittag–Leffler function, we give estimates for the solution, which imply the continuous dependency of solutions on initial values and source terms. Next we will deduce the Lipschitz stability of the solution to (1.1)–(1.3) with respect to α_j, q_j $(j = 1, \ldots, m)$ and diffusion coefficients. Finally, for the long-time asymptotic behavior, we employ the Laplace transform in time to show that the decay rate as $t \to \infty$ is exactly $t^{-\alpha_m}$, where α_m is the minimum order of Caputo derivatives in time.

The rest of this paper is organized as follows. The main results concerning problem (1.1)–(1.3) are collected in Section 2: Theorems 2.1,2.2,2.3 assert the well-posedness and Theorem 2.4 is concerned with the long-time asymptotic behavior of the solution. The proofs of the well-posedness results are given in Section 3 on the basis of several properties of the multinomial Mittag–Leffler functions. Due to the difference of techniques, the asymptotic behavior is proved in Section 4. Next, the proofs of technical lemmata on the multinomial Mittag–Leffler functions are postponed to Section 5. Finally, concluding remarks are given in Section 6.

2. Main results

In this section, we state the main results obtained in this paper. More precisely, we give a priori estimates for the solution u to (1.1)–(1.3) with respect to the initial value (Theorem 2.1), the source term (Theorem 2.2), and Lipschitz continuous dependency of the solutions on coefficients and orders (Theorem 2.3), and we describe the asymptotic behavior of the solution in Theorem 2.4.

To this end, we first fix some general settings and notations. Let $L^2(\Omega)$ be a usual L^2-space with the inner product $(\cdot,\cdot)$ and $H_0^1(\Omega)$, $H^2(\Omega)$ denote the Sobolev spaces (see, e.g., [2]). The elliptic operator L is defined for $f \in \mathcal{D}(-L) := H^2(\Omega) \cap H_0^1(\Omega)$ as

$$Lf(x) = \sum_{i,j=1}^{d} \partial_j(a_{ij}(x)\partial_i f(x)) + c(x)f(x), \quad x \in \Omega,$$

where $a_{ij} = a_{ji}$ $(1 \leqslant i, \; j \leqslant d)$ and $c \leqslant 0$ in $\overline{\Omega}$. Moreover, it is assumed that $a_{ij} \in C^1(\overline{\Omega})$, $c \in C(\overline{\Omega})$ and there exists a constant $\delta > 0$ such that

$$\delta \sum_{i=1}^{d} \xi_i^2 \leqslant \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j, \quad \forall x \in \overline{\Omega}, \; \forall (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d.$$