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Second-order explicit difference schemes for the space fractional advection diffusion equation

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ABSTRACT

In this paper, two kinds of explicit second order difference schemes are developed to solve the space fractional advection diffusion equation. The discretizations of fractional derivatives are based on the weighted and shifted Grünwald difference operators developed in [Meerschaert and Tadjeran, J.Comput.Appl.Math. 172 (2004) 65–77; Tian et al., arXiv:1201.5949; Li and Deng, arXiv:1310.7671]. The stability of the presented difference schemes are discussed by means of von Neumann analysis. The analysis shows that the presented numerical schemes are both conditionally stable. The necessary conditions of stability is discussed. Finally, the results of numerical experiments are given to illustrate the performance of the presented numerical methods.

1. Introduction

In this paper, we discuss the explicit difference schemes for the following fractional advection diffusion equation

$$u_t(x,t) + v u_x(x,t) = d_x\left(\kappa_1 {}_aD_x^\alpha + \kappa_2 {}_xD_b^\alpha\right)u(x,t) + f(x,t), \tag{1}$$

where $u(x,t)$ is the concentration of a solute at a point x at time t, $f(x,t)$ is the source term, $v(>0)$ is the advection coefficient, $d_x(>0)$ is the diffusion coefficient. The parameters $\kappa_1 \geqslant 0$ and $\kappa_2 \geqslant 0$ are skewed parameters that control the bias of the dispersion [1]. And $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, ${}_aD_x^\alpha$ and ${}_xD_b^\alpha$ are the left and right Riemann–Liouville fractional derivatives of order $\alpha(1 < \alpha < 2)$, respectively, defined by [18,20,33]

$${}_aD_x^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_a^x (x-\xi)^{1-\alpha} u(\xi,t)d\xi \tag{2}$$

and

$${}_xD_b^\alpha u(x,t) = \frac{(-1)^2}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_x^b (\xi-x)^{1-\alpha} u(\xi,t)d\xi. \tag{3}$$

Eq. (1) arise naturally in many applications, such heat conduction with anomalous diffusion, nonlocal reactive flow in porous media and non-Fickian flow of fluids in porous media, see [1,16,3,8]. Over the last decade, there has been an interest in numerical methods or algorithms for solving the space fractional advection diffusion equation (1). The finite difference

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approximation is relatively easy to implement, there some progress in the numerical solution of fractional advection diffusion equations in the past decade, see [9,14,12,15,27,5,11,6].

In recent years, high order accurate difference schemes and fast algorithms for space fractional partial differential equations with Riemann–Liouville fractional derivatives attract many authors' interesting. Base on the Toeplitz-like structure of the difference matrix presented in [14], Wang et al. [31] presented a $N\log^2 N$ finite difference algorithm for fractional diffusion equations. Later, Pang and Sun [19] proposed a multigrid method to solve the discretized linear system of the fractional diffusion equation. Two main techniques are popular to get the higher order accurate approximations for the fractional derivatives. The first one is using higher order numerical quadrature formula to approximate the Riemann–Liouville fractional derivatives. In the literature, by introducing the piecewise linear interpolation, Sousa and Li presented a second order discretization for the Riemann–Liouville fractional derivatives and established an unconditionally stable weighted average finite difference method for one-dimensional fractional diffusion equation in [24]. Based on linear interpolation, Deng and Chen [6] provided a second order finite difference scheme to solve three-dimensional fractional advection diffusion equation. Another way of designing higher order approximation is to recombine the Grünwald–Letnikov difference operators or modify the generation function of weights in Grünwald–Letnikov derivatives [7,18]. Ortigueira in his work [17] gave the "fractional centered derivative" to approximate the Riesz fractional derivative with second order accuracy, and this method was used by Çelik and Duman to approximate fractional diffusion equation with the Riesz fractional derivative in a finite domain [2]. For more application and extension of "fractional centered derivative" one can see the recent works [26,22] and references, therein. By combining the shifted Grünwald–Letnikov derivatives given by Meerschaert and Tadjeran [14], Tian et al. [29] proposed some second order difference approximations, called by weighted and shifted Grünwald–Letnikov difference(WSGD) approximations, to the Riemann–Liouville fractional derivatives. A family of second accurate difference schemes are established for the fractional diffusion equations by those approximations. The key issue of the WSGD approximation is combining the distinct shifted Grünwald–Letnikov formulae with their corresponding weights. Motivated by this idea, in our recent work [13], we introduced some possible extensions of the second order WSGD approximations presented in the previous work [29]. Based on weighting the shifted Grünwald–Letnikov derivatives by multi-parameters, we derive a new family of second order difference operators for Riemann–Liouville fractional derivatives. Furthermore, using the new second order difference discretizations, we designed two kinds of implicit difference schemes for the space fractional advection diffusion equation. Comparing to the implicit difference scheme, the discretized linear system of explicit difference scheme is not concerned with the inverse matrix. It is popular for the equations which advection term appears. However, it is stable under some conditions. So it is important to analysis the stability of explicit difference scheme [10,28,21]. So far, limit works are reported to discuss the stability of explicit difference schemes for space fractional advection diffusion equation besides Sousa's works [23,25]. With the help of von Neumann analysis method, we analyze the stability of the proposed numerical methods. The analysis show that explicit central-WSGD scheme and Lax–Wendroff-WSGD schemes are stable under some certain restriction on the time and spatial steps.

The remaining part of this article is organized as follows. In Section 2, we briefly introduce the second order WSGD operators for the Riemann–Liouville fractional derivatives. We then present two kinds of difference schemes for the space-fractional advection diffusion equation with second order accuracy. The stability and convergence of the method are proved in Section 3. Finally, in Section 4, we present two numerical examples to demonstrate the performance of the proposed numerical schemes. Numerical results support our analysis and show the performance of the proposed numerical schemes.

2. Second order approximation schemes

We shall design difference formulations for the advection diffusion equation (1) with the initial condition

$$u(x,0) = g(x), \quad x \in (a,b)$$

and the Dirichlet boundary conditions

$$u(a,t) = \phi_a(t), \quad u(b,t) = \phi_b(t), \quad t \in (0,T].$$

And if $\kappa_1 \neq 0$, then $\phi_a(t) \equiv 0$ and $\kappa_2 \neq 0$, then $\phi_b(t) \equiv 0$. In order to construct difference schemes for the advection diffusion equation (1) with above initial-boundary conditions, we first introduce the discretizations of the Riemann–Liouville fractional derivatives. In fact, many different definitions of fractional derivatives, such as the Grünwald–Letnikov derivative, Riemann–Liouville derivative, the Caputo derivative, and other modified definitions are introduced for the practical application [18]. Among these definitions, the Grünwald–Letnikov derivatives is more suitable for numerical approximation. It is usual used to approach the Riemann–Liouville derivative. For a function $u(x) \in C[a,b]$, the relation between left Grünwald–Letnikov derivatives of $u(x)$ with $u(a) = 0$ and left Riemann–Liouville derivatives of function $u(x)$ gives [18]

$${}_aD_x^{\alpha}u(x) = \frac{1}{h^{\alpha}}\sum_{k=0}^{\left[\frac{x-a}{h}\right]} w_k^{(\alpha)} u(x-kh) + O(h).$$

And the relation between right Grünwald–Letnikov of $u(x)$ with $u(b) = 0$ and right Riemann–Liouville derivatives of function $u(x)$ gives [18]