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Bifurcation from interval and positive solutions of the three-point boundary value problem for fractional differential equations

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ABSTRACT

This paper investigates the existence of positive solutions for fractional differential equations with the three-point boundary value conditions. The main tools used here are bifurcation techniques and topological degree theory. Two examples to illustrate the applications of main results are also given.

1. Introduction

Fractional differential equations can be extensively applied to the various physics, mechanics, chemistry and engineering etc., for more details on fractional calculus theory, one can see the monographs of Podlubny [1], Kilbas [2], Diethelm [3] and Zhou [4].

It is worth remarking that initial value problems or linear boundary value problems for fractional differential equations have been given considerable attention by many authors, see [5–17] and the references therein. The discussions in these papers are mainly concerned with the existence and multiplicity of positive solutions by using different methods, such as the fixed point theorem, topology degree theory, the lower and upper solutions method and critical point theory.

In [11], Li et al. consider the fractional boundary value problem (BVP for short) of the following form

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad t \in J := [0,1], \\ u(0) = 0, \quad D_{0+}^{\beta}u(1) = aD_{0+}^{\beta}u(\xi), \end{cases} \tag{1.1}$$

where D_{0+}^{α} and D_{0+}^{β} are the Riemann–Liouville fractional derivatives of order $1 < \alpha \leqslant 2$, $0 \leqslant \beta < 1$, $0 \leqslant a \leqslant 1$, $\xi \in (0,1)$, $a\xi^{\alpha-\beta-2} \leqslant 1-\beta$, $\alpha-\beta-1 \geqslant 0$ and $f : J \times [0,\infty) \to [0,\infty)$ satisfies the Carathodory condition. By using some fixed-point theorems, they established some sufficient criteria of existence and multiplicity results for positive solutions.

In this paper, we use a powerful method due to Liu [18] to investigate the problem (1.1) and to obtain the existence of positive solutions under weaker conditions, here, f is not necessarily linearizable at (0,0) and infinity. Liu's method is commonly known as bifurcation techniques and topology degree theory. The method has been highly useful for proving existence of positive solutions for initial and boundary value problem for differential equations, see [18–23].

This paper is organized as follows. In Section 2, we give some notations, recall some concepts and preparation results. In Section 3, we study the existence of solutions for the BVP (1.1) based on bifurcation techniques. Finally, in Section 4, two examples are considered to illustrate the applicability of our abstract results.

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2. Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper, let us recall some definitions of fractional calculus, for more details see [2].

Definition 2.1 [2]. The fractional integral of order γ with the lower limit zero for a function $f : [0,\infty) \to \mathbf{R}$ is defined as

$$I_t^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \quad t > 0, \ \gamma > 0$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 [2]. The Riemann–Liouville derivative of order γ with the lower limit zero for a function $f : [0,\infty) \to \mathbf{R}$ can be written as

$$D_{0+}^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, \quad t > 0,$$

where $n = [\gamma] + 1$ and $[\gamma]$ denotes the integer part of γ.

Lemma 2.3 [2]. *Let $\gamma > 0$, then the differential equation $D_{0+}^{\gamma} u(t) = 0$ has solutions*

$$u(t) = c_0 + c_1 t^{\gamma-1} + c_2 t^{\gamma-2} + \cdots + c_{n-1} t^{\gamma-n},$$

where $c_i \in \mathbf{R}, \ i = 0,1,2,\ldots,n, \ n = [\gamma] + 1$.

Lemma 2.4 [24]. *Let V be a real Banach space. Let $\widetilde{G} : \mathbf{R} \times V \to V$ be completely continuous such that $\widetilde{G}(\lambda, 0) = 0$ for all $\lambda \in \mathbf{R}$. There exist $a, b \in \mathbf{R}(a < b)$ such that $u = 0$ is an isolated solution of the equation*

$$u - \widetilde{G}(\lambda, u) = 0, \quad u \in V \tag{2.1}$$

for $\lambda = a$ and $\lambda = b$, where $(a,0), (b,0)$ are not bifurcation points of (2.1). Furthermore, assume that

$$\deg(I - \widetilde{G}(a,\cdot), B_r(0), 0) \neq \deg(I - \widetilde{G}(b,\cdot), B_r(0), 0),$$

where $B_r(0)$ is an isolating neighborhood of the trivial solution. Let

$$T = \overline{\{(\lambda,u) : (\lambda,u) \text{ is a solution of } (2.1) \text{ with } u \neq 0\}} \cup ([a,b] \times \{0\}).$$

Then there exists a connected component $\mathcal{C}$ of T containing $[a,b] \times \{0\}$ in $\mathbf{R} \times V$, and either

(i) $\mathcal{C}$ is unbounded in $\mathbf{R} \times V$ or
(ii) $\mathcal{C} \cap [(\mathbf{R} \setminus [a,b]) \times \{0\}] \neq \emptyset$.

Lemma 2.5 [25]. *Let V be a real Banach space. Let $\widetilde{G} : \mathbf{R} \times V \to V$ be completely continuous, and there exist $a, b \in \mathbf{R}(a < b)$ such that the solution of (2.1) are a priori bounded in V for $\lambda = a$ and $\lambda = b$, that is, there exists an $R > 0$ such that*

$$\widetilde{G}(a,u) \neq u \neq \widetilde{G}(b,u),$$

for all u with $\|u\| \geqslant R$. Furthermore, assume that

$$\deg(I - \widetilde{G}(a,\cdot), B_R(0), 0) \neq \deg(I - \widetilde{G}(b,\cdot), B_R(0), 0)$$

for $R > 0$ sufficiently large. Then there exists a closed connected set $\mathcal{C}$ of solutions of (2.1) that is unbounded in $[a,b] \times V$, and either

(i) $\mathcal{C}$ is unbounded in λ direction or
(ii) there exists an interval $[c,d]$ such that $(a,b) \cap (c,d) = \emptyset$ and $\mathcal{C}$ bifurcates from infinity in $[c,d] \times V$.

Lemma 2.6 [26]. *Let Ω be a bounded open set of real Banach space $E, A : \bar{\Omega} \to E$ be completely continuous. If there exists $y_0 \in E, \ y_0 \neq \theta$ such that*

$$x \in \partial\Omega, \quad \tau \geqslant 0 \Rightarrow x - Ax \neq \tau y_0.$$

Then

$$\deg(I - A, \Omega, \theta) = 0.$$