Applied Mathematics and Computation 257 (2015) 487–497

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Convergence of solutions of mixed stochastic delay differential equations with applications

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ARTICLE INFO

Keywords:
Mixed stochastic differential equation
Stochastic delay differential equation
Convergence of solutions
Fractional Brownian motion
Vanishing delay
Euler approximation

ABSTRACT

The paper is concerned with a mixed stochastic delay differential equation involving both a Wiener process and a γ-Hölder continuous process with $\gamma > 1/2$ (e.g. a fractional Brownian motion with Hurst parameter greater than 1/2). It is shown that its solution depends continuously on the coefficients and the initial data. Two applications of this result are given: the convergence of solutions to equations with vanishing delay to the solution of equation without delay and the convergence of Euler approximations for mixed stochastic differential equations. As a side result of independent interest, the integrability of solution to mixed stochastic delay differential equations is established.

1. Introduction

In this paper we consider a multidimensional mixed stochastic delay differential equation

$$dX(t) = a(t, X_t)dt + b(t, X_t)dW(t) + c(t, X_t)dZ(t), \quad t \in [0, T]. \tag{1}$$

In this equation the coefficients $a,\ b,\ c$ depend on the past $X_s = \{X(s+u), u \in [-r, 0]\}$ of the process X, and the initial condition is thus given by $X(t) = \eta(t),\ t \in [-r, 0]$, where $\eta : [-r, 0] \to \mathbb{R}$ is some function. Eq. (1) is driven by two random processes: a standard Wiener process W and a process Z, whose trajectories are γ-Hölder continuous with $\gamma > 1/2$. The process Z is usually a long memory process, e.g. a fractional Brownian motion B^H with the Hurst parameter $H > 1/2$.

Itô stochastic delay differential equations, i.e. those with $c = 0$, were investigated in many articles, see [9,13] and references therein. Fractional stochastic delay differential equations, in which $b = 0$ and $Z = B^H$, were considered in only few papers. For $H > 1/2$, the existence and uniqueness of solution under different sets of assumptions was established in [1–5,8]. In the case $H > 1/3$, the existence and uniqueness of solution was shown in [14] for coefficients of the form $f(X(t), X(t-r_1), X(t-r_2), \ldots)$.

The existence and uniqueness of solution to Eq. (1) was established in [15], where also the finiteness of moments was shown under the additional assumption that the coefficient b is bounded. Mixed equations without delay were considered in articles [6,7,11,12,16,17].

In this article we prove that if the coefficients and initial conditions of mixed stochastic delay differential equations converge, then their solutions converge uniformly in probability. We give two applications of this result. First we show that when the delay vanishes, solutions of mixed stochastic delay differential equations converge uniformly in probability to a solution of equation without delay. Then we establish the uniform convergence of Euler approximations for mixed stochastic

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http://dx.doi.org/10.1016/j.amc.2015.01.019
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differential equation towards its solution. We also extend the results of [16] about the integrability of solution to (1), dropping the assumption that b is bounded.

The paper is organized as follows. Section 2 provides necessary information about pathwise stochastic integration and describes the notation used in the article. Section 3 contains the main result of the article about convergence of solutions to mixed stochastic delay differential equations with convergent coefficients and initial conditions. It also contains a result of independent interest about the integrability of solution to mixed stochastic delay differential equation. Section 4 is devoted to applications of the convergence results. In SubSection 4.1, a sequence of mixed stochastic delay differential equations is considered with delay horizon converging to zero, and it is shown that their solutions converge to a solution of equation without delay. In SubSection 4.2 it is proved that the Euler approximations for mixed stochastic differential equation (without delay) converge to its solution, as the mesh of partition goes to zero. Appendix contains an auxiliary result about convergence of solutions to Itô stochastic delay differential equations with random coefficients, which is used in the proof of main theorem.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t, t \geqslant 0\}, P)$ be a standard stochastic basis.

Throughout the article, $|\cdot|$ will denote the absolute value of a real number, the Euclidean norm of a vector, or the operator norm of a matrix. The symbol C will denote a generic constant, whose value may change from one line to another. To emphasize its dependence on some parameters, we will put them into subscripts.

We will need the notion of generalized fractional Lebesgue–Stieltjes integral. Below we give only basic information on the integral, further details and proofs can be found in [18].

Let $f,\ g : [a,b] \to \mathbb{R},\ \alpha \in (0,1)$. Define the forward and backward fractional Riemann–Liouville derivatives

$$(D^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha\int_a^x \frac{f(x)-f(u)}{(x-u)^{1+\alpha}}du\right),$$
$$\left(D^{1-\alpha}_{b-}g\right)(x) = \frac{e^{i\pi\alpha}}{\Gamma(\alpha)}\left(\frac{g(x)}{(b-x)^{1-\alpha}} + (1-\alpha)\int_x^b \frac{g(x)-g(u)}{(u-x)^{2-\alpha}}du\right),$$

where $x \in (a,b)$.

The generalized Lebesgue–Stieltjes integral is defined as

$$\int_a^b f(x)dg(x) = e^{i\pi\alpha}\int_a^b (D^{\alpha}_{a+}f)(x)\left(D^{1-\alpha}_{b-}g_{b-}\right)(x)dx$$

provided the integral in the right-hand side exists. For functions $f,\ g : [a,b] \to \mathbb{R}$ and a number $\alpha \in (0,1)$ define

$$\|f\|_{1,\alpha;[a,b]} = \int_a^b \left(\frac{|f(a)|}{(t-a)^{\alpha}} + \int_a^t \frac{|f(t)-f(s)|}{(t-s)^{1+\alpha}}ds\right)dt, \tag{2}$$
$$\|g\|_{0,\alpha;[a,b]} = \sup_{a\leqslant s\leqslant t\leqslant b}\left(\frac{|g(t)-g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(u)-g(s)|}{(u-s)^{2-\alpha}}du\right). \tag{3}$$

Note that only the first expression defines a norm, the second defines a seminorm. The following fact is evident: if $\|f\|_{1,\alpha;[a,b]} < \infty$ and $\|g\|_{0,\alpha;[a,b]} < \infty$, then the generalized Lebesgue–Stieltjes integral is well defined and admits an estimate

$$\left|\int_a^b f(x)dg(x)\right| \leqslant \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\|f\|_{1,\alpha;[a,b]}\|g\|_{0,\alpha;[a,b]}. \tag{4}$$

We will also need an estimate in terms of Hölder norms. Namely, if $f \in C^{\lambda}[a,b],\ g \in C^{\mu}[a,b]$ with $\lambda \in (0,1),\ \mu \in (0,1)$ and $\lambda + \mu > 1$, then the generalized Lebesgue–Stieltjes integral is well defined and coincides with the Riemann–Stieltjes integral. Moreover, the Young–Love inequality holds:

$$\left|\int_a^b f(s)dg(s)\right| \leqslant C_{\lambda,\mu}\|g\|_{a,b,\mu}\left(\|f\|_{a,b,\infty} + \|f\|_{a,b,\lambda}(b-a)^{\lambda}\right)(b-a)^{\mu},$$

where $\|f\|_{a,b,\infty} = \sup_{x\in[a,b]}|f(x)|$ is the supremum norm on $[a,b]$, and

$$\|f\|_{a,b,\lambda} = \sup_{a\leqslant x<y\leqslant b}\frac{|f(y)-f(x)|}{(y-x)^{\lambda}}$$

is the Hölder seminorm.