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Fractional-order total variation image denoising based on proximity algorithm

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ABSTRACT

The fractional-order total variation(TV) image denoising model has been proved to be able to avoid the "blocky effect". However, it is difficult to be solved due to the non-differentiability of the fractional-order TV regularization term. In this paper, the proximity algorithm is used to solve the fractional-order TV optimization problem, which provides an effective tool for the study of the fractional-order TV denoising model. In this method, the complex fractional-order TV optimization problem is solved by using a sequence of simpler proximity operators, and therefore it is effective to deal with the problem of algorithm implementation. The final numerical procedure is given for image denoising, and the experimental results verify the effectiveness of the algorithm.

1. Introduction

Image denoising is historically one of the oldest topic in image processing and is still an important preprocessing step for many applications [1]. It is a typical inverse problem, which estimates the original image describing a real scene from the observed image of the same scene. There are many methods for image denoising, and we are interested in the total variation(TV) method in this paper. Since the work of Rudin–Osher–Fatemi [2], the TV minimization problems have arisen in many image processing applications for regularizing inverse problems in which piecewise constant processed image was expected to be achieved, and a multitude of methods were proposed to solve this problem [3–9]. The ROF model can achieve a good trade-off between edge preservation and noise removal, but it tends to cause the blocky effect due to the fact that it produces a piecewise constant solution. Many methods have been proposed to deal with this problem. In 2000, You and Kaveh [10] introduced a fourth-order partial differential equation (PDE)-based denoising model in which the regularized solution was obtained by solving the minimization of potential function of second-order derivative of the image. Chan, Marquina and Mulet [11] presented an improved high-order TV model, constructed by adding a nonlinear fourth order diffusive term to the Euler–Lagrange equations of TV model. In 2003, a new method for image smoothing based on a fourth-order PDE model was proposed by Lysaker, Lundervold and Tai (LLT), which was tested on a broad range of real medical magnetic resonance images [12]. Lysaker and Tai [13] proposed a denoising technique, which combined the TV filter with the fourth-order PDE filter. Bai and Feng [14] presented a new class of fractional-order anisotropic diffusion equations for noise removal, which can be seen as the generalizations of the second-order and the fourth-order anisotropic diffusion

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equations. The discrete Fourier transform was used to implement the numerical algorithm. In this paper, we focus on the fractional-order TV model.

Fractional calculus is a rapidly growing mathematical discipline, which provides an important tool for nonlocal field theories [16,15]. Recently, the fractional-order total variation (TV) models have been proposed for image denoising [17–21], image inpainting [22], image super-resolution [23] and motion estimation [24]. However, it is difficult to minimize the fractional-order TV regularized functional due to a lack of differentiability. A lot of efforts have been devoted to obtain the fast numerical schemes and overcome the non-differentiability of the model. In [24], the solution of fractional-order variational model was obtained by solving the associated fractional-order Euler–Lagrange equations. Zhang et al. [17,18] developed an alternation projection algorithm for the fractional-order multi-scale variational model and proposed an efficient condition of the convergence of the algorithm. The majorization–minimization (MM) algorithm was used to solve the fractional-order TV image denoising model in [20]. In [21], the primal–dual algorithm was used to solve the saddle-point problem of the fractional-order TV model, which was able to guarantee $O(1/N^2)$ convergence rate. In this paper, we will use the proximity algorithm to solve the fractional-order TV denoising model, which provides a novel framework for the study of the fractional-order TV model. An important advantage of this framework is to provide a convenient analysis for the convergence of fractional-order TV denoising method.

This paper is organized as follows: Section 1 introduces the prior works and our motivation. In Section 2, the fractional-order total variation denoising model is described, and the corresponding discrete model is given. Based on this, the proximity algorithm is used to solve this fractional-order TV minimization problem, and the final numerical implementation is presented for image denoising. Experimental evaluation is presented in Section 3 and the paper is concluded in Section 4.

2. Fractional-order TV denoising model and proximity algorithm

2.1. Model description

We denote the given image by the function $f : \Omega \to \mathbb{R}$, where Ω is a subset of $\mathbb{R}^2$ representing the image domain. The typical variational formulation of total variation (TV) denoising problem is of the form

$$\min_u \int_\Omega \frac{1}{2}(u-f)^2 + \mu|\nabla u| \mathrm{d}\Omega, \tag{1}$$

where μ is the regularisation parameter which controls the degree of smoothing, and $u : \Omega \to \mathbb{R}$ is the desired clean image.

In this paper, we focus on the fractional-order TV denoising model which is described by

$$\min_u \int_\Omega \frac{1}{2}(u-f)^2 + \mu|\nabla^\alpha u| \mathrm{d}\Omega, \tag{2}$$

where $\nabla^\alpha u = (\partial_x^\alpha u, \partial_y^\alpha u)^T$ and $|\nabla^\alpha u| = |\partial_x^\alpha u| + |\partial_y^\alpha u|$. The parameter α is a real number. When $\alpha = 1$, Eq. (2) is equal to Eq. (1). So, the fractional-order TV model can be seen as the generalization of the typical integer order TV model.

Consider that the given image has $n_1 \times n_2$ pixels. The pixels should be distributed on a regular grid with distance $h_x = 1$ and $h_y = 1$ between two pixels in the respective direction. The discrete version of the data term is given by

$$\int_\Omega (u-f)^2 \approx \sum_{i,j} (u_{i,j} - f_{i,j})^2, \tag{3}$$

where (i,j) denotes the coordinates at the points we have discretized the integral. For smoothing term, we obtain the following discretization

$$\int_\Omega |\nabla^\alpha u|\Omega \approx \sum_{i,j} |\nabla_x^\alpha u_{i,j}| + |\nabla_y^\alpha u_{i,j}|. \tag{4}$$

From Grünwald–Letnikov fractional derivative definition [25], the finite fractional-order difference can be obtained by

$$\nabla_x^\alpha u_{i,j} = \sum_{k=0}^{K-1} c_k^{(\alpha)} u_{i-k,j}, \quad \nabla_y^\alpha u_{i,j} = \sum_{k=0}^{K-1} c_k^{(\alpha)} u_{i,j-k}, \tag{5}$$

where $c_k^{(\alpha)} = (-1)^k C_k^\alpha, C_k^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}$ denotes the generalized binomial coefficient and $\Gamma(x)$ is the Gamma function. In additional, the coefficients $c^{(\alpha)}$ can also be obtained recursively from

$$c_0^{(\alpha)} = 1, \quad c_k^{(\alpha)} = \left(1 - \frac{\alpha+1}{k}\right) c_{k-1}^{(\alpha)}, \quad k = 1, 2, \ldots. \tag{6}$$

When $\alpha = 1, c_k^1 = 0$ for $k > 1$ and Eq. (5) is the first-order difference as usual.

For simplicity, we reorder the image matrices u and f row-wise into the vector p and g, and associate the (i,j) element of the two-dimensional structure with the element $(j-1)n_1 + i$ of the vector structure, $p_{(j-1)n_1+i} = u_{i,j}$ and $g_{(j-1)n_1+i} = f_{i,j}$. We

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