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Mellin transforms of generalized fractional integrals and derivatives

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ABSTRACT

We obtain the Mellin transforms of the generalized fractional integrals and derivatives that generalize the Riemann–Liouville and the Hadamard fractional integrals and derivatives. We also obtain interesting results, which combine generalized $\delta_{r,m}$ operators with generalized Stirling numbers and Lah numbers. For example, we show that $\delta_{1,1}$ corresponds to the Stirling numbers of the 2nd kind and $\delta_{2,1}$ corresponds to the unsigned Lah numbers. Further, we show that the two operators $\delta_{r,m}$ and $\delta_{m,r}, r, m \in \mathbb{N}$, generate the same sequence given by the recurrence relation.

$$S(n,k) = \sum_{i=0}^{r} (m + (m-r)(n-2) + k - i - 1)_{r-i} \binom{r}{i} S(n-1, k-i), \quad 0 < k \leqslant n,$$

with $S(0,0) = 1$ and $S(n,0) = S(n,k) = 0$ for $n > 0$ and $1 + min\{r,m\}(n-1) < k$ or $k \leqslant 0$. Finally, we define a new class of sequences for $r \in \{\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\}$ and in turn show that $\delta_{\frac{1}{2},1}$ corresponds to the generalized Laguerre polynomials.

1. Introduction

The Fractional Calculus (FC) is the generalization of the classical calculus to arbitrary orders. The history of FC goes back to seventeenth century, when in 1695 the derivative of order $\alpha = \frac{1}{2}$ was described by Leibniz. Since then, the new theory turned out to be very attractive to mathematicians as well as biologists, economists, engineers and physicists [27]. In [46], Samko et al. provide a comprehensive treatment of the subject. Several different derivatives were studied: Riemann–Liouville, Caputo, Hadamard, Erdélyi–Kober, Grunwald–Letnikov and Riesz are just a few to name [24,46]. In [19], the author provides an extended reference to some of these and other fractional derivatives.

In fractional calculus, the fractional derivatives are defined via fractional integrals [24,28,35,46]. According to the literature, the Riemann–Liouville fractional derivative, hence the Riemann–Liouville fractional integral has played a major roles in FC [46]. The Caputo fractional derivative has also been defined via the Riemann–Liouville fractional integral [46]. Another important fractional operator is the Hadamard operator [14,36]. In [7], Butzer, et al. obtained the Mellin transforms of the Hadamard integral and differential operators. The interested reader is referred to, for example, [5–7,21–24,36,46] for further properties of those operators.

In [18], the author introduces a new fractional integral, which generalizes the Riemann–Liouville and the Hadamard integrals into a single form. The interested reader is referred, for example, to [4,12,15,17,20,29–34,37,40] for further results

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on these and similar operators. In [19], the author introduces a new fractional derivative which generalizes the Riemann–Liouville and the Hadamard fractional derivatives to a single form. The present work is primarily devoted to the Mellin transforms of the generalized fractional operators developed in [18,19].

The paper is organized as follows. In the next section, we give definitions and some basic properties of the fractional integrals and derivatives of various types. In Section 3, we develop the Mellin transforms of the generalized fractional integrals and derivatives. Further, we investigate the relationship between the Mellin transform operator and the generalized Stirling numbers of the 2^{nd} kind. Finally, we introduce a new class of sequences.

2. Definitions

The *Riemann–Liouville fractional integrals* $I^\alpha_{a+}f$ and $I^\alpha_{b-}f$ of order $\alpha \in \mathbb{C}, (Re(\alpha) > 0)$ are defined by [46],

$$\left(I^\alpha_{a+}f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau) d\tau; \quad x > a. \tag{2.1}$$

and

$$\left(I^\alpha_{b-}f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau-x)^{\alpha-1} f(\tau) d\tau; \quad x < b, \tag{2.2}$$

respectively, where $\Gamma(\cdot)$ is the Gamma function. These integrals are called the *left-sided* and *right-sided fractional integrals*, respectively. When $\alpha = n \in \mathbb{N}$, the integrals (2.1) and (2.2) coincide with the n-fold integrals [24, chap. 2]. The *Riemann–Liouville fractional derivatives* (RLFD) $D^\alpha_{a+}f$ and $D^\alpha_{b-}f$ of order $\alpha \in \mathbb{C}$, $(Re(\alpha) \geqslant 0)$ are defined by [46],

$$\left(D^\alpha_{a+}f\right)(x) = \left(\frac{d}{dx}\right)^n \left(I^{n-\alpha}_{a+}f\right)(x) \quad x > a, \tag{2.3}$$

and

$$\left(D^\alpha_{b-}f\right)(x) = \left(-\frac{d}{dx}\right)^n \left(I^{n-\alpha}_{b-}f\right)(x) \quad x < b, \tag{2.4}$$

respectively, where $n = [Re(\alpha)] + 1$, which is sometimes denoted also by the ceiling function, $\lceil Re(\alpha) \rceil$, when there is no room for confusion. Here $[\cdot]$ represents the integer part. For simplicity, from this point onwards, except in few occasions, we consider only the *left-sided* integrals and derivatives. The interested reader may find more detailed information about the *right-sided* integrals and derivatives in the references, for example in [24,46].

The next type that we elaborate in this paper is the *Hadamard Fractional integral* [14,24,46] given by,

$$\mathbf{I}^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log\frac{x}{\tau}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau}; \quad Re(\alpha) > 0, x > a \geqslant 0. \tag{2.5}$$

The corresponding *Hadamard fractional derivative* of order $\alpha \in \mathbb{C}, Re(\alpha) > 0$ is given by,

$$\mathbf{D}^\alpha_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)} \left(x\frac{d}{dx}\right)^n \int_a^x \left(\log\frac{x}{\tau}\right)^{n-\alpha+1} f(\tau) \frac{d\tau}{\tau}; \quad x > a \geqslant 0. \tag{2.6}$$

where $n = \lceil Re(\alpha) \rceil$.

In 1940, important generalizations of the Riemann–Liouville fractional operators were introduced by Erdélyi and Kober [9]. *Erdélyi–Kober*-type fractional integral and differential operators [9,10,24,25,46,49] are defined by,

$$\left(I^\alpha_{a+;\,\rho,\,\eta}f\right)(x) = \frac{\rho x^{-\rho(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho\eta+\rho-1} f(\tau)}{(x^\rho - \tau^\rho)^{1-\alpha}} d\tau \tag{2.7}$$

for $x > a \geqslant 0, Re(\alpha) > 0$ and

$$\left(D^\alpha_{a+;\,\rho,\,\eta}f\right)(x) = x^{-\rho\eta} \left(\frac{1}{\rho x^{\rho-1}} \frac{d}{dx}\right)^n x^{\rho(n+\eta)} \left(I^{n-\alpha}_{a+;\,\rho,\,\eta+\alpha}f\right)(x) \tag{2.8}$$

for $x > a, Re(\alpha) \geqslant 0, \rho > 0$, respectively. When $\rho = 2, a = 0$, the operators are called *Erdélyi–Kober* operators. When $\rho = 1, a = 0$, they are called *Kober-Erdélyi* or *Kober operators* [24, p.105]. Extensive treatment of these operators can be found especially in [49]. Further generalizations can be found, for example in [24,46].

In [18], the author introduces an another generalization to the Riemann–Liouville and the Hadamard fractional integral and also provided existence results and semi-group properties. In [19], the author showed that both the Riemann–Liouville and the Hadamard fractional derivatives can be represented by a single fractional derivative operator. These new operators have been defined on the following extended Lebesgue space.