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# A fast semi-implicit difference method for a nonlinear two-sided space-fractional diffusion equation with variable diffusivity coefficients

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ABSTRACT

In this paper, we derive a new nonlinear two-sided space-fractional diffusion equation with variable coefficients from the fractional Fick's law. A semi-implicit difference method (SIDM) for this equation is proposed. The stability and convergence of the SIDM are discussed. For the implementation, we develop a fast accurate iterative method for the SIDM by decomposing the dense coefficient matrix into a combination of Toeplitz-like matrices. This fast iterative method significantly reduces the storage requirement of $O(n^2)$ and computational cost of $O(n^3)$ down to $n$ and $O(n \log n)$, where $n$ is the number of grid points. The method retains the same accuracy as the underlying SIDM solved with Gaussian elimination. Finally, some numerical results are shown to verify the accuracy and efficiency of the new method.



## 1. Introduction

In the past few decades, anomalous diffusion modeled using governing equations involving space- and/or time-fractional operators has generated great interest. From the point of view of probability, fractional calculus describe an anomalous diffusion phenomenon, in which a cloud of particles spreads in a different manner than traditional diffusion (see [1] for details). The fractional diffusion model provides a more adequate and accurate description of memory and hereditary properties of anomalous transport process in heterogenous porous media, and has been successfully applied to simulate processes in biological systems [2–6], hydrology [7–9], image processing [10], and physics [11].

In practical problems, the diffusion coefficient is usually space or time dependent. However, few papers have focused on the variable-coefficient space fractional diffusion equation in conservative form. Therefore, it is the aim of this paper to derive a more generalized fractional diffusion model with variable coefficients in conservative form. According to the principle of conservation of mass, the equation of continuity in one-dimensional form is given by

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial Q(x,t)}{\partial x} - f(u,x,t) = 0, \tag{1}$$

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where $Q(x,t)$ denotes the diffusion flux, $f(u,x,t)$ is the nonlinear source term, and $u(x,t)$ is the distribution function of the diffusing quantity. We modified the classical Fick's law by

$$Q(x,t) = -C(x)\frac{\partial}{\partial x}\int_a^x k_+(x,\xi)u(\xi,t)d\xi - D(x)\frac{\partial}{\partial x}\int_x^b k_-(x,\xi)u(\xi,t)d\xi, \tag{2}$$

where $C(x)$ and $D(x)$ are the nonnegative diffusion coefficients, $C(x)$ decreases monotonically with respect to $x$ and $D(x)$ increases monotonically with respect to $x$ in the domain $[a,b]$; $k_+(x,\xi), k_-(x,\xi)$ are the kernel functions defined by

$$\begin{cases} k_+(x,\xi) = \frac{1}{\Gamma(1-\alpha)}(x-\xi)^{-\alpha}, & \text{for} \quad a \leqslant \xi \leqslant x, \\ k_-(x,\xi) = \frac{1}{\Gamma(1-\alpha)}(\xi-x)^{-\alpha}, & \text{for} \quad x \leqslant \xi \leqslant b, \end{cases} \tag{3}$$

with $0 < \alpha < 1$. The combination of Eqs. (1) and (2) results in the following conservative form of a nonlinear two-sided space-fractional diffusion model with variable diffusivity coefficients:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(C(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} - D(x)\frac{\partial^\alpha u(x,t)}{\partial(-x)^\alpha}\right) + f(u,x,t), \quad a \leqslant x \leqslant b, \quad 0<\alpha<1, \quad t>0, \tag{4}$$

where the operators $\frac{\partial^\alpha}{\partial x^\alpha}, \frac{\partial^\alpha}{\partial(-x)^\alpha}$ are the left and right Riemann–Liouville fractional derivatives (see [1,12,13]) defined by

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_a^x \frac{u(\xi,t)}{(x-\xi)^{\alpha+1-n}}d\xi, \tag{5}$$

$$\frac{\partial^\alpha u(x,t)}{\partial(-x)^\alpha} = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_x^b \frac{u(\xi,t)}{(\xi-x)^{\alpha+1-n}}d\xi, \tag{6}$$

respectively, for $n-1 \leqslant \alpha < n$, where $n$ is an integer. In this paper, we focus on the construction of an unconditionally stable and effective semi-implicit difference scheme for (4) subjected to the boundary and initial conditions

$$u(a,t) = u(b,t) = 0, \quad 0 \leqslant t \leqslant T, \tag{7}$$
$$u(x,0) = \phi(x), \quad a \leqslant x \leqslant b. \tag{8}$$

Anh and Leonenko [14] presented a spectral representation of the mean-square solution of the fractional kinetic equation with random initial condition. Generally, numerical solution techniques are preferred when dealing with fractional models since the analytical solutions are only available for a few simple cases. During the last decade, extensive research has been carried out on the development of efficient numerical solutions for fractional partial differential equations, including finite difference methods [15–20], the finite volume method [21], the finite element method [22–24], and the spectral method [25,26]. In contrast to numerical methods for the integer-order partial differential equation, which usually generates a banded coefficient matrix, the finite difference discretization of the space-fractional model results in a linear system with a full, or dense, coefficient matrix. Normally speaking, owing to the non-local nature of fractional operators, the traditional approach (Gaussian elimination method, for instance) to solve the resulting linear system requires an $O(n^2)$ storage and an $O(n^3)$ computational cost for a problem of size $n$. Therefore, from the computational point of view, efficient and effective numerical methods that can significantly reduce the amount of CPU time are of high importance. Wang et al. [27] developed an $O(n\log^2 n)$ fast finite difference method for a fractional diffusion equation by carefully analyzing the special structure of the coefficient matrix, and then extended this technique to derive a fast alternating-direction finite difference method for the two-dimensional fractional diffusion model [28]. Wang et al. [29] constructed a fast conjugate gradient squared method (FCGS) by decomposing the coefficient matrix into a combination of sparse and Toeplitz-like matrices. Moroney [30] presented a fast Poisson preconditioner for a Jacobian-free Newton–Krylov method to solve the nonlinear space-fractional diffusion equations.

In this paper, we develop a semi-implicit difference method (SIDM) for the new nonlinear two-sided space-fractional diffusion equation with variable coefficients and prove the unconditional stability and convergence of the SIDM. Based on the observation that the coefficient matrix of the SIDM can be decomposed into a combination of sparse and Toeplitz-like dense matrices, we develop a fast bi-conjugate gradient stabilized method (FBi-CGSTAB) for the proposed difference strategy by exploiting the Toeplitz-like structure of the coefficient matrix.

The remainder of this paper is organized as follows. In Section 2, we construct a semi-implicit difference method (SIDM) for a space-fractional diffusion model (4). In Section 3, we discuss the consistency and solvability, stability, and convergence of the SIDM. In Section 4, we present a fast bi-conjugate gradient stabilized method that significantly reduces the computational complexity from $O(n^3)$ to $O(n\log n)$ and the memory requirement from $O(n^2)$ to $O(n)$ for the model of size of $n$. Finally, we carry out numerical experiments to verify the performance of our method by comparing consumed CPU time with the regular method in Section 5.