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# A general form of the generalized Taylor's formula with some applications

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ABSTRACT

In this article, a new general form of fractional power series is introduced in the sense of the Caputo fractional derivative. Using this approach some results of the classical power series are circulated and proved to this fractional power series, whilst a new general form of the generalized Taylor's formula is also obtained. Some applications including fractional power series solutions for higher-order linear fractional differential equations subject to given nonhomogeneous initial conditions are provided and analyzed to guarantee and to confirm the performance of the proposed results. The results reveal that the new fractional expansion is very effective, straightforward, and powerful for formulating the exact solutions in the form of a rapidly convergent series with easily computable components.



## 1. Introduction

Fractional calculus is the study of theory and applications of integrals and derivatives of arbitrary order. This branch of mathematical analysis, extensively investigated in the recent years, has emerged as an effective and powerful tool for the mathematical modeling of several engineering and scientific phenomena. Therefore, it is necessary to have some mathematical apparatus and tools in order to understand this huge concept. The theory of fractional calculus goes back to Leibniz, Liouville, Riemann, Grunwald, and Letnikov [1–4]. The interest in the fractional differential equations (FDEs) has been growing continually during the last few years because of numerous applications. In a short period of time the list of applications becomes long, for example, it includes the quantum mechanics, chaotic dynamics, material sciences, mechanics of fractal and complex media, physical kinetics, plasma physics, electromagnetic theory, non-Hamiltonian mechanics, and long-range interaction [5–13].

Classical power series (CPS) have become a fundamental tool in the study of elementary functions and also other not so elementary as can be checked in any book of analysis. They have been widely used in the computational science obtaining an easy approximation of functions [14]. In physics, chemistry, and many other sciences this power expansion has allowed scientists to make an approximate study of many systems, neglecting higher order terms around the equilibrium point. This is a fundamental tool to linearize a problem which guaranties easy analysis [15–20].

Fractional power series (FPS) is also becoming an essential tool in the study of elementary functions, particularly in the fractional calculus approach. The ordinary Taylor's formula, which is a CPS, has been generalized by many authors [21–23]. Recently, Odibat and Shawagfeh [24] presented a new generalized Taylor's formula as follows:

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$$f(t)=\sum_{m=0}^{n}\frac{D_{t_0}^{m\alpha}f(t_0)}{\Gamma(m\alpha+1)}(t-t_0)^{m\alpha}+\frac{D_{t_0}^{(n+1)\alpha}f(\xi)}{\Gamma((n+1)\alpha+1)}(t-t_0)^{(n+1)\alpha},\quad 0<\alpha\leqslant 1,\quad t_0<t\leqslant b. \tag{1.1}$$

El-Ajou and et. al. [25] introduced a new proof and a new form to the generalized Taylor's formula as follows:

$$f(t)=\sum_{n=0}^{\infty}\frac{D_{t_0}^{n\alpha}f(t_0)}{\Gamma(n\alpha+1)}(t-t_0)^{n\alpha},\quad m-1<\alpha\leqslant m,\quad t_0<t\leqslant b. \tag{1.2}$$

In this article, we dealt with a FPS in general which is a generalization to the CPS. Important theorems related to the CPS have been generalized and constructed to the FPS in the sense of the Caputo fractional derivatives. A general form of the generalized Taylor's formula of Eq. (1.1) has been derived for $0\leqslant m-1<\alpha\leqslant m, m\in\mathbb{N}$. This new form of the Taylor's formula has been used to construct FPS solutions to the following class of higher-order linear nonhomogeneous FDEs:

$$D_{t_0}^{\alpha}y(t)+\lambda_1 t^{m-1}y^{(m-1)}(t)+\lambda_2 t^{m-2}y^{(m-2)}(t)+\cdots+\lambda_{m-2}ty'(t)+\lambda_{m-1}y(t)+\lambda_m=g(t),\ t\geqslant t_0, \tag{1.3}$$

subject to the following nonhomogeneous initial conditions:

$$y^{(i)}(t_0)=\delta_i,\quad i=0,1,2,\ldots,m-1, \tag{1.4}$$

where $\lambda_i,\delta_i\in\mathbb{R}, m\in\mathbb{N}, 0\leqslant m-1<\alpha\leqslant m$, and $g(t)$ can be expanded in the general form of the generalized Taylor's formula of Eqs. (3.7) and (3.8). Throughout this paper $D_{t_0}^{m\alpha}=D_{t_0}^{\alpha}\cdot D_{t_0}^{\alpha}\cdots D_{t_0}^{\alpha}$($m$-times) is the Caputo fractional derivative of order $m\alpha$, $\mathbb{N}$ the set of natural numbers, $\mathbb{R}$ the set of real numbers, and $\Gamma$ is the Gamma function.

The outline of the paper is as follows. In the next section, we present some necessary definitions and preliminary results that will be used in our work. In Section 3, some definitions and theorems related to FPS are mentioned and proved. In Section 4, series solutions of linear FDEs are produced and discussed with two computational applications. This article ends in Section 5 with some concluding remarks.

## 2. Excerpts of fractional calculus theory

The contents of this section are basic in some sense, for reader's convenience, we present some necessary definitions from fractional calculus theory and preliminary results. There are various definitions of fractional integration and fractional differentiation, such as Grünwald-Letnikov's definition, Riemann–Liouville's definition, Caputo's definition, and generalized function approach [1–4]. For the purpose of this paper, Caputo's definition of fractional differentiation will be used, taking the advantage of Caputo's approach that the initial conditions for FDEs with Caputo's derivatives take on the traditional form, similar to those for integer order differential equations [1].

Next, some necessary definitions and essentials results from fractional calculus theory are presented, where these definitions and results are collected from the mentioned references. Firstly, we shall introduce a modified fractional differential operator $D_s^{\alpha}$ proposed by Caputo in his work on the theory of viscoelasticity.

**Definition 2.1.** A real function $f(x), x>0$ is said to be in the space $C_{\mu}, \mu\in\mathbb{R}$ if there exists a real number $p>\mu$ such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C[0,\infty)$, and it is said to be in the space $C_{\mu}^{n}$ if $f^{(m)}(x)\in C_{\mu}, m\in\mathbb{N}$.

**Definition 2.2.** The Riemann–Liouville fractional integral operator of order $\alpha\geqslant 0$ of $f(x)\in C_{\mu}, \mu\geqslant -1$ is defined as

$$J_s^{\alpha}f(x)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_s^x (x-t)^{\alpha-1}f(t)dt, & x>t>s\geqslant 0, \alpha>0,\\ f(x), & \alpha=0.\end{cases} \tag{2.1}$$

**Definition 2.3.** The Caputo fractional derivative of order $\alpha>0$ of $f\in C_{-1}^{n}, m\in\mathbb{N}$ is defined as

$$D_s^{\alpha}f(x)=\begin{cases}J_s^{m-\alpha}f^{(m)}(x), & x>s\geqslant 0, m-1<\alpha<m,\\ \frac{d^m f(x)}{dx^m}, & \alpha=m.\end{cases} \tag{2.2}$$

For some certain properties of the operator $D_s^{\alpha}$, it is obvious that, when $\gamma>-1, x>s\geqslant 0, \alpha\geqslant 0$, and $C\in\mathbb{R}$, we have $D_s^{\alpha}(x-s)^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}(x-s)^{\gamma-\alpha}$ and $D^{\alpha}C=0$, while on the other aspect as well, properties of the operator $J_s^{\alpha}$ can be summarized shortly in the form of the following: for $f\in C_{\mu}, \mu\geqslant -1, \gamma\geqslant -1, \alpha,\beta\geqslant 0$, and $C\in\mathbb{R}$, we have $J_s^{\alpha}C=\frac{C}{\Gamma(\alpha+1)}(x-s)^{\alpha}$, $J_s^{\alpha}J_s^{\beta}f(x)=J_s^{\alpha+\beta}f(x)=J_s^{\beta}J_s^{\alpha}f(x)$, and $J_s^{\alpha}(x-s)^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(x-s)^{\alpha+\gamma}$.

**Theorem 2.1.** *If $m-1<\alpha\leqslant m$, $f\in C_{\mu}^{m}$, $m\in\mathbb{N}$, and $\mu\geqslant -1$, then $D_s^{\alpha}J_s^{\alpha}f(x)=f(x)$ and $J_s^{\alpha}D_s^{\alpha}f(x)=J_s^{n}D_s^{n}f(x)=f(x)-\sum_{j=0}^{m-1}f^{(j)}(s^{+})\frac{(x-s)^j}{j!}$, where $x>s\geqslant 0$.*