Applied Mathematics and Computation 256 (2015) 83–93

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Convergence of the scheme with weights for the numerical solution of a heat conduction equation with delay for the case of variable coefficient of heat conductivity

Andrey Lekomtsev, Vladimir Pimenov *

Department of Computational Mathematics, Ural Federal University, 19 Mira street, Ekaterinburg 620002, Russia

ARTICLE INFO

Keywords:
Parabolic equations
Delay
Scheme with weights
Variable coefficient of heat conductivity

ABSTRACT

One-dimensional parabolic equations with delay effects in the time component for the case of variable coefficient of heat conductivity are considered. The scheme with weights is constructed for the numerical solution of these equations. The order of approximation error for the constructed scheme, stability, and order of convergence are investigated.

1. Introduction

In the simulation of dynamics processes often simultaneously two effects are found: distribution of parameters in space and heredity in time [1,2]. Such equations include diffusion equation with delay of general type, constant or variable, concentrated or distributed. Numerical methods for solving such equations were considered in many papers, see for example [3–8].

Options of the method of lines, in which discretization is carried out only on state variables, reduce problem to the numerical solving systems of functional differential equations [4,5]. However at such discretization there are the stiff systems which rigidity increases at increase in number of points of discretization on a state.

In a number of works grid methods were investigated, see for example [6] and the bibliography to this work. In these works the main idea consists in replacement of an initial problem its discrete analog and introduction of intermediate interpolation space. At such approach there are following problems: solvability of the received system of the difference equations, stability of algorithms and their convergence. Thus, since the effect of delay can enter nonlinearly, the nonlinear system of the difference equations of large dimension turns out.

In the present paper we consider the quasilinear parabolic equation with the effect of delay entering only in inhomogeneous term. By linear part algorithms known for the parabolic equations are used (schemes with weights). For the accounting of delay along with idea of carrying out interpolation of discrete prehistory for creation of the differential model adequate to the initial equation the idea of extrapolation of prehistory is used. Extrapolation allows to avoid the nonlinear equations in spaces of large dimension. As a result on each temporary layer we obtain a linear system of the equations which can be effectively solved by the sweep method. This approach allows us to construct effective algorithms that can be the basis of the corresponding program packages. Justification of convergence of these algorithms is carried out by application both the general theory of differential schemes [9], and the general technique of research of difference schemes of the solution of functional differential equations [10,11]. The last technique uses the ideas of work [12] developed for the ordinary differential equations. In works [13,14] this technique was applied to research of numerical methods of the solution of the parabolic

* Corresponding author.
E-mail addresses: lekom@tt66.ru (A. Lekomtsev), Vladimir.Pimenov@usu.ru (V. Pimenov).

http://dx.doi.org/10.1016/j.amc.2014.12.149

equations in case of constant coefficients, with one and two spatial variables and with effect of delay. In the present paper existence of dependence from time of coefficient of heat conductivity demanded modification of this general technique of research of difference schemes of the solution of the functional differential equations.

The structure of paper is the following. In the second section the formulation of the problem is provided and the main assumptions become.

In the third section discretization of a task is carried out. Interpolation's and extrapolation's constructions are for this purpose given and the family of schemes with weights is described. The concept of the residual of a method without interpolation is entered and its order is studied.

The fourth section contains the main theoretical base: the description of the general difference scheme with effect of heredity. Concepts of a temporary grid are entered; discrete model in each timepoint as element of finite-dimensional normed space, prehistory of discrete model, interpolation space, obvious step-by-step formula, starting values, function of exact values. In a step-by-step formula the operator of transition is allocated, its properties define stability of the scheme. The main statement the theorem of an order of convergence of the scheme which depends on an order residual with interpolation is proved.

In the fifth section the embedding of the algorithm described in the third section is carried out to this scheme. Connection of an order of residual without interpolation, an order of interpolation and an order of residual with interpolation is established. The main attention is paid to a conclusion of the condition guaranteeing stability. As a result of the stated results, the theorem of orders of convergence of the methods described in this paper turns out. The order of convergence of the methods is quadratic in time and space steps.

Results of numerical experiments are given in the last section: test example with the distributed delay and a model example from population dynamics with constant delay. The equations contain variable coefficients in both examples.

2. Problem statement and main assumptions

Let us consider a one-dimensional heat conduction equation with delay in the form:

$$\frac{\partial u}{\partial t} = a^2(t)\frac{\partial^2 u}{\partial x} + f(x, t, u(x, t), u_t(x, \cdot)), \tag{1}$$

here, $x \in [0, X] \subset \mathbb{R}^1$ and $t \in [t_0, \theta] \subset \mathbb{R}^1$ are independent space and time variables, respectively; $u(x, t)$ is the required function; $u_t(x, \cdot) = \{u(x, t+s), -\tau \leqslant s < 0\}$ is the prehistory of the required function by the time t; and τ is the value of delay.

Let the initial conditions

$$u(x, t) = \varphi(x, t), \quad x \in [0, X],\ t \in [t_0 - \tau, t_0] \tag{2}$$

and the boundary conditions

$$u(0, t) = u(X, t) = 0, \quad t \in [t_0, \theta] \tag{3}$$

be given.

Problem (1)–(3) is the simplest boundary-value problem for the one-dimensional heat conduction equation with delay effect in the general form for the case of variable coefficient of heat conductivity. We assume that coefficient $a^2(t)$ satisfies the following condition

$$0 < c_1 \leqslant a^2(t) \leqslant c_2, \quad t \in [t_0, \theta]. \tag{4}$$

We assume also that the functions $a(t), \varphi(x, t)$ and the functional f are such that problem (1)–(3) has a unique solution $u(x, t)$ in the classical sense. Moreover, we assume that the function $u(x, t)$ has the smoothness required in the reasonings below.

We denote by $Q = Q[-\tau, 0)$ the set of functions $u(s)$ that are piecewise continuous on the half-open interval $[-\tau, 0)$ with a finite number of points of discontinuity of the first kind and right continuous at the points of discontinuity. In addition, the functions $u(s)$ have a finite left-hand limit at zero. We define the norm of a function on Q by the relation $\|u(\cdot)\|_Q = \sup_{-\tau \leqslant s < 0}|u(s)|$. We additionally assume that the functional $f(x, t, u, v)$ is given on $[0, X] \times [t_0, \theta] \times R \times Q$ and is Lipschitz with respect to the last two arguments; i.e., there exists a constant L_f such that, for all $x \in [0, X], t \in [t_0, \theta]$, $u^1 \in R, u^2 \in R, v^1(\cdot) \in Q$, and $v^2(\cdot) \in Q$, the following inequality holds:

$$|f(x, t, u^1, v^1(\cdot)) - f(x, t, u^2, v^2(\cdot))| \leqslant L_f(|u^1 - u^2| + \|v^1(\cdot) - v^2(\cdot)\|_Q).$$

3. Scheme with weights

Let us divide the interval $[0, X]$ of variation of the space variable into parts with step $h = X/N$, introducing the points $x_i = ih, i = 0, \dots, N$, and the interval $[t_0, \theta]$ of variation of the time variable into parts with step $\Delta > 0$, introducing the points $t_k = t_0 + k\Delta, k = 0, \dots, M$. We assume that the value $\tau/\Delta = m$ is an integer.