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A combinatorial identity involving gamma function and Pochhammer symbol

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ABSTRACT

In this paper we establish an identity that relates Pochhammer symbol and the ratio of gamma functions. The identity is derived using the Mellin series representation for the solution of a general algebraic equation.

1. Introduction

The gamma function, first introduced by Euler in the 1720s when he was solving the problem of finding a function of a continuous variable $x > 0$ that equals $x!$ when $x = 1, 2, 3, \ldots$,

$$x! = \Gamma(x+1) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0 \tag{1}$$

is one of the most famous and important function in the theory of special functions, since all the hypergeometric series are defined in terms of the shifted factorials (rising factorial). It is well-known that the gamma function can be analytically continued to the whole complex plane excluding non-positive integers.

The gamma function is of great importance in pure mathematics, for example, number theory and mathematical analysis, and has various applications in other branches of science, such as probability theory, mathematical statistics, physics, just to mention a few areas only. As a consequence, the gamma function has been investigated intensively by many authors. In particular, in recent years there has been considerable interest in inequalities concerning Euler's gamma function and numerous interesting inequalities for the gamma function have been published and are known in mathematical literature (see, e.g., [5,7,10] and references therein). Much effort has also been invested to estimate gamma and polygamma functions and to establish new, increasingly better bounds (see [4,8] and references therein), as well as to study the asymptotic behavior of the ratio of gamma functions $\Gamma(x)$ for large values of x, since it is important and useful in several contexts of mathematical analysis, such as the study of integrals of the Mellin–Barnes type and the investigation of the asymptotic behavior of confluent hypergeometric functions [9]. In addition, a number of identities involving ratios of gamma functions is widely presented and can be found in the literature.

The goal of this paper is to establish a combinatorial identity involving partitions of integers for the ratio of gamma functions. These results are presented in Section 2.

In this paper, $x!$ is understood as the gamma function (1), that is for $x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$ we define

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$$x! := \Gamma(x+1), \tag{2}$$

where Γ denotes the gamma function.

We also use the following notations for the falling factorials

$$[x]_k = x(x-1)\dots(x-k+1) = \frac{\Gamma(x+1)}{\Gamma(x-k+1)}, \quad k \geqslant 1$$

and for the rising factorials (Pochhammer's symbol)

$$(x)_k = x(x+1)\dots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)}, \quad k \geqslant 1,$$

and $[x]_0 = 1, (x)_0 = 1$.

2. Main results

The main result of this paper is the following proposition.

Theorem 2.1. *For $k > 0$ and $p > q > 0, k, p, q \in \mathbb{N}$, we have*

$$\frac{\Gamma\left(\frac{1+qk}{p}\right)}{\Gamma\left(\frac{1-(p-q)k}{p}\right)} = \frac{1-(p-q)k}{p} \sum_{k-1=m_2+\dots+(k-1)m_k} \frac{(-1)^{k-1+m_2+\dots+m_k}}{p^{k-1+m_2+\dots+m_k}} \times \frac{(k-1+m_2+\dots+m_k)!}{m_2!\dots m_k!} \left(\frac{(q)_2-(-(p-q))_2}{2!}\right)^{m_2} \times \dots \left(\frac{(q)_{k-1}-(-(p-q))_{k-1}}{(k-1)!}\right)^{m_{k-1}} \left(\frac{(q)_k-(-(p-q))_k}{k!}\right)^{m_k}, \tag{3}$$

where $(-s)_r = (-1)^r[s]_r$, and the sum runs over all partitions of $(k-1)$.

Proof. We first consider a general algebraic equation

$$y^n + x_1 y^{n_1} + x_2 y^{n_2} + \dots + x_l y^{n_l} - 1 = 0, \quad n > n_1 > \dots > n_l > 0. \tag{4}$$

The principal solution of (4), that is, the solution which satisfies the condition $y(0) = 1$, is expressed via Mellin series as follows

$$y = y(x_1,\dots,x_l) = 1 + \frac{1}{n} \sum_{\substack{j_1+\dots+j_l \geqslant 1 \\ j_1,\dots,j_l \geqslant 0}} (-1)^{j_1+\dots+j_l} \frac{x_1^{j_1}\dots x_l^{j_l}}{j_1!\dots j_l!} \times \left[\frac{1}{n} - 1 + \frac{n_1}{m} j_1 + \dots + \frac{n_l}{m} j_l\right]_{j_1+\dots+j_l-1}. \tag{5}$$

For $l = 1$ the Eq. (4) reduces to a trinomial equation

$$y^p + xy^q - 1 = 0, \; p > q > 0. \tag{6}$$

From (5) for $l = 1$ it follows that

$$y = 1 + \frac{1}{p}\sum_{k>0} (-1)^k \left[\frac{1}{p} - 1 + \frac{q}{p}k\right]_{k-1} \frac{x^k}{k!}.$$

For the falling factorial $\left[\frac{1}{p} - 1 + \frac{q}{p}k\right]_{k-1}$ we have

$$\left[\frac{1}{p} - 1 + \frac{q}{p}k\right]_{k-1} = \frac{\left(\frac{q}{p}k + \frac{1}{p} - 1\right)!}{\left(\frac{q}{p}k + \frac{1}{p} - 1 - k + 1\right)!} = \frac{\left(\frac{qk+1-p}{p}\right)!}{\left(\frac{qk+1-kp}{p}\right)!},$$

and hence

$$y = 1 + \frac{1}{p}\sum_{k>0} \frac{(-1)^k}{k!} \frac{\left(\frac{qk+1-p}{p}\right)!}{\left(\frac{qk+1-kp}{p}\right)!} x^k. \tag{7}$$

Since, according to (2), here $x!$ is the gamma function, the Mellin series for the principal solution of trinomial Eq. (6) is

$$y = 1 + \frac{1}{p}\sum_{k>0} \frac{(-1)^k}{k!} \frac{\Gamma\left(\frac{1+qk}{p}\right)}{\Gamma\left(\frac{p+1-(p-q)k}{p}\right)} x^k. \tag{8}$$