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On the convergence of King–Werner-type methods of order $1+\sqrt{2}$ free of derivatives

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ABSTRACT

We present a semilocal and local convergence analysis of some efficient King–Werner-type methods of order $1+\sqrt{2}$ free of derivatives in a Banach space setting. Numerical examples are presented to illustrate the theoretical results.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is Fréchet-differentiable operator defined on a convex subset of a Banach space X with values in a Banach space Y.

Many problems in Computational Sciences and other disciplines can be brought in a form like (1.1) using mathematical modeling [2,5,12]. The solutions of these equations can be rarely be found in closed form. That is why most solution methods for these equations are usually iterative. The practice of Numerical Functional Analysis for finding such solutions is essentially connected to Newton-like methods [1–14]. Newton's method converges quadratically to $x^\star$ if the initial guess is close to enough to the solution. Iterative methods of convergence order higher than two such as Chebyshev–Halley-type methods [1–3,7–14] require the evaluation of the second Fréchet-derivative, which is very expensive in general. However, there are integral equations where the second Fréchet-derivative is diagonal by blocks and inexpensive [1,2,5] or for quadratic equations, the second Fréchet-derivative is constant. Moreover, in some applications involving stiff systems, high order methods are useful. That is why it is important to study high-order methods.

The study about convergence of iterative procedures is normally centered on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure. While the local analysis is based on the information around a solution, to find estimates of the radii of convergence balls.

In particular, Werner in [13,14] studied a method originally proposed by King [8] defined by:

Given $x_0, y_0 \in D$, let

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$$x_{n+1} = x_n - F'\left(\tfrac{x_n+y_n}{2}\right)^{-1} F(x_n),$$
$$y_{n+1} = x_{n+1} - F'\left(\tfrac{x_n+y_n}{2}\right)^{-1} F(x_{n+1}), \tag{1.2}$$

for each $n = 0, 1, 2, \ldots$ and $X = \mathbb{R}^i, Y = \mathbb{R}$ where i is a whole number. The local convergence analysis is based on assumptions of the form:

(H_0) There exists $x^\star \in D$ such that $F(x^\star) = 0$;
(H_1) $F \in C^{2,a}(D), a \in (0, 1]$;
(H_2) $F'(x)^{-1} \in L(Y, X)$ and $\|F'(x)^{-1}\| \leqslant \Gamma$;
(H_3) The Lipschitz condition

$$\|F'(x) - F'(y)\| \leqslant L_1\|x - y\|,$$

holds for each $x, y \in D$;
(H_4) The Lipschitz condition

$$\|F''(x) - F''(y)\| \leqslant L_{2,a}\|x - y\|^a,$$

holds for each $x, y \in D$;
(H_5) $U(x^\star, b) \subseteq D$ for

$$b = \min\{b_0, \rho_0, \rho_1\},$$

where

$$b_0 = \frac{2}{L_1\Gamma},$$

and ρ_0, ρ_1 solve the system [14]

$$Bv^{1+a} + Aw = 1,$$
$$2Av^2 + Avw = w,$$
$$A = \frac{1}{2}\Gamma L_1, \quad B = \frac{\Gamma L_{2,a}}{4(a+1)(a+2)}.$$

The convergence order was shown to be $1 + \sqrt{2}$. However, there are cases where, e.g. (H_4) is violated. For an example, define function $f : [-1, 1] \to (-\infty, \infty)$ by

$$f(x) = x^2 \ln x^2 + c_1x^2 + c_2x + c_3, \quad f(0) = c_3,$$

where c_1, c_2, c_3 are given real numbers. Then, we have that $\lim_{x\to 0} x^2 \ln x^2 = 0, \lim_{x\to 0} x \ln x^2 = 0, f'(x) = 2x \ln x^2 + 2(c_1 + 1)x + c_2$ and $f''(x) = 2(\ln x^2 + 3 + c_1)$. Then, function f does not satisfy (H_4).

Recently, McDougall et al. in [9] studied a two-step method:

Given $x_0 \in D$, let

$$y_0 = x_0,$$
$$x_1 = x_0 - F'\left(\tfrac{x_0+y_0}{2}\right)^{-1} F(x_0),$$
$$y_n = x_n - F'\left(\tfrac{x_{n-1}+y_{n-1}}{2}\right)^{-1} F(x_n),$$
$$x_{n+1} = x_n - F'\left(\tfrac{x_n+y_n}{2}\right)^{-1} F(x_n), \tag{1.3}$$

for each $n = 1, 2, \ldots$ and $X = Y = \mathbb{R}$. Comparing method (1.3) with the King–Werner method (1.2), one easily see that (1.3) is just the King–Werner-type method with repeated initial points, that is, $x_0 = y_0$. Notice that the initial predictor step in (1.3) is a Newton step based on the estimated derivative. The re-use of the derivative means that the evaluations of the y_n values in (1.3) essentially come for free, which then enables the more appropriate value of the derivative to be used in the corrector step in (1.3). Method (1.3) was also shown to be of order $1 + \sqrt{2}$ in [9]. Notice also that the convergence of method (1.3) was given under the (H) conditions in [13,14].

In the present paper we study the convergence analysis of method defined for $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = x_n - A_n^{-1} F(x_n),$$
$$y_{n+1} = x_{n+1} - A_n^{-1} F(x_{n+1}), \tag{1.4}$$

where x_0, y_0 are initial points, $A_n = [x_n, y_n; F]$ and $[x, y; F]$ denotes a divided difference of order one for operator F at points $x, y \in D$ [2,5,7,12] satisfying