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A functional generalization of the interpolation problem

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ABSTRACT

Given two linearly independent functions f_1 and f_2, we generalize the interpolating problem to the space $\pi_n(f_1,f_2)$ spanned by the basis $\left\{f_1^{n-k}f_2^k\right\}_{k=0}^n$. We show that this problem has a unique solution and represent this solution by a functional analogue of the Lagrange formula. We also give a similar generalization of Hermite interpolation.

1. Introduction

Let f_1 and f_2 be two linearly independent functions on $[a,b]$. Then the functions $\phi_k^n = f_1^{n-k}f_2^k,\ k = 0,1,\ldots,n$ are linearly independent on $[a,b]$ (see [1]). We consider interpolation problem in the space

$$\pi_n(f_1,f_2) = \text{span}\{\phi_0^n, \phi_1^n, \ldots, \phi_n^n\}.$$

Using the barycentric coordinates of a point on the planar parametric curve $P(t) = (f_1(t),f_2(t)),\ a \leqslant t \leqslant b$ satisfying $d(x_1,x_2) = f_1(x_1)f_2(x_2) - f_1(x_2)f_2(x_1)$ that never vanishes for distinct $x_1,x_2 \in [a,b]$, we establish the solution. The barycentric coordinates of a point $P(x) = (f_1(x),f_2(x)),\ a \leqslant x \leqslant b$ on this curve relative to the endpoints of the arc of $P(t)$ from $P(a) = (f_1(a),f_2(a))$ to $P(b) = (f_1(b),f_2(b))$ are the solutions $\alpha(x,a,b),\ \beta(x,a,b)$ of the system

$$\alpha f_1(a) + \beta f_1(b) = f_1(x),$$
$$\alpha f_2(a) + \beta f_2(b) = f_2(x).$$

Solving this system of equations gives $\alpha(x,a,b) = \frac{d(x,b)}{d(a,b)}$ and $\beta(x,a,b) = \frac{d(a,x)}{d(a,b)}$. It is shown in [1] that every element from the space $\pi_n(f_1,f_2)$ can be represented in terms of the barycentric coordinates.

Note that if $f_1(x) = 1$ and $f_2(x) = x$, then $d(x_1,x_2) = x_2 - x_1$ never vanishes for distinct $x_1,x_2 \in \mathbb{R}$ and the space $\pi_n(1,x)$ is space of the polynomials of degree n. If $f_1(x) = \cos x$ and $f_2(x) = \sin x$, then $d(x_1,x_2) = \sin(x_2 - x_1)$ never vanishes for distinct $x_1,x_2 \in [a,b]$, where $b - a < \pi$. It is given in [2] that

$$\pi_n(\cos x, \sin x) = \begin{cases} \text{span}\{\sin x, \cos x, \sin 3x, \ldots, \sin nx, \cos nx\}, & n \text{ is odd}, \\ \text{span}\{1, \sin 2x, \cos 2x, \ldots, \sin nx, \cos nx\}, & n \text{ is even}. \end{cases} \tag{1}$$

Since $d(x_1,x_2)$ never vanishes, the barycentric coordinates

$$l_1(x) = \frac{d(x,x_2)}{d(x_1,x_2)} \quad \text{and} \quad l_2(x) = \frac{d(x_1,x)}{d(x_1,x_2)}$$

form a uni-solvent system. A system of k functions $l_1,l_2,\ldots,l_k$ defined on a point set S is called uni-solvent on S if

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$$|l_i(x_j)| \neq 0$$

holds for every selection of k distinct points lying in S. (See [3].)

We proceed in the following fashion. In Section 2 we investigate the interpolation problem in the space $\pi_n(f_1,f_2)$ and find an error term. Section 3 gives the representation of the interpolating function in both the Lagrange form and the Hermite form.

2. Existence and uniqueness

A classical approach to solve the polynomial interpolation problem is to solve a system of linear equations that involve the Vandermonde matrix. We now generalize this.

Theorem 2.1. *Given $n+1$ distinct points $x_0,x_1,\dots,x_n$ and corresponding values $y_0,y_1,\dots,y_n$. Then there exists a unique function $g(x)\in\pi_n(f_1,f_2)$ for which*

$$g(x_i)=y_i,\quad i=0,1,\dots,n.$$

Proof. Let $g(x)=a_0\phi_0^n(x)+a_1\phi_1^n(x)+\cdots+a_n\phi_n^n(x)$ be the interpolating function. Imposing the interpolating conditions lead to a system of $n+1$ linear equations in $n+1$ unknowns:

$$a_0\phi_0^n(x_i)+a_1\phi_1^n(x_i)+\cdots+a_n\phi_n^n(x_i)=y_i,$$

$i=0,1,\dots,n$. It remains to show that this system of equations has unique solution, that is the determinant

$$V_{f_1f_2}(x_0,\dots,x_n)=\begin{vmatrix}\phi_0^n(x_0) & \cdots & \phi_{n-1}^n(x_0) & \phi_n^n(x_0)\\ \phi_0^n(x_1) & \cdots & \phi_{n-1}^n(x_1) & \phi_n^n(x_1)\\ \vdots & \vdots & \vdots & \vdots\\ \phi_0^n(x_n) & \cdots & \phi_{n-1}^n(x_n) & \phi_n^n(x_n)\end{vmatrix} \tag{2}$$

is nonzero. This determinant may be viewed as a functional analogue of the Vandermonde determinant. Evaluating $V_{f_1f_2}$ is similar to the classical case (see [3]). Consider the function

$$V(x)=V_{f_1f_2}(x_0,\dots,x_{n-1},x)=\begin{vmatrix}\phi_0^n(x_0) & \cdots & \phi_{n-1}^n(x_0) & \phi_n^n(x_0)\\ \vdots & \vdots & \vdots & \vdots\\ \phi_0^n(x_{n-1}) & \cdots & \phi_{n-1}^n(x_{n-1}) & \phi_n^n(x_{n-1})\\ \phi_0^n(x) & \cdots & \phi_{n-1}^n(x) & \phi_n^n(x)\end{vmatrix}.$$

It follows from the expansion of the above determinant by its last row that $V_{f_1f_2}(x_0,\dots,x_{n-1},x)\in\pi_n(f_1,f_2)$. Substituting $x=x_i,\ i=0,1\dots,n-1$ gives two identical rows in the determinant, that is; $V(x)$ has n zeros at $x_i,\ i=0,1\dots,n-1$. Hence we may write

$$V_{f_1f_2}(x_0,x_1,\dots,x)=Cd(x_0,x)d(x_1,x)\cdots d(x_{n-1},x), \tag{3}$$

where C depends only on $x_0,x_1,\dots,x_{n-1}$. Comparing the coefficients of $(f_2(x))^n$ on both sides of the Eq. (3) yields

$$\begin{vmatrix}f_1(x_0)\phi_0^{n-1}(x_0) & \cdots & f_1(x_0)\phi_{n-1}^{n-1}(x_0)\\ \vdots & \vdots & \vdots\\ f_1(x_{n-1})\phi_0^{n-1}(x_{n-1}) & \cdots & f_1(x_{n-1})\phi_{n-1}^{n-1}(x_{n-1})\end{vmatrix}=Cf_1(x_0)\cdots f_1(x_{n-1}).$$

Every element in the jth row of the above determinant has a factor $f_1(x_{j-1}),\ j=1,2,\dots,n$. Thus we obtain

$$f_1(x_0)\cdots f_1(x_{n-1})V_{f_1f_2}(x_0,\dots,x_{n-1})=Cf_1(x_0)\cdots f_1(x_{n-1}).$$

Cancelling the terms gives the following recurrence

$$V_{f_1f_2}(x_0,\dots,x_n)=V_{f_1f_2}(x_0,\dots,x_{n-1})d(x_0,x_n)\cdots d(x_{n-1},x_n). \tag{4}$$

Since $V_{f_1f_2}(x_0,x_1)=d(x_0,x_1)$ and the points x_i, for $i=0,1,\dots n$ are distinct, by repeated application we obtain

$$V_{f_1f_2}(x_0,\dots x_n)=\prod_{j<i}^{n}d(x_j,x_i)\neq 0.\qquad\square \tag{5}$$

The following theorem gives the error between $f(x)$ and its interpolating function $g(x)$.