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On generalized of certain retarded nonlinear integral inequalities and its applications in retarded integro-differential equations

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ABSTRACT

In this paper, some new nonlinear retarded integral inequalities of Gronwall–Bellman–Pachpatte type are investigated. Some applications are also presented to illustrate the usefulness of some of our results in estimation of solution of certain retarded integro-differential equations with the initial conditions.

1. Introduction

The integral inequalities which provide explicit bounds on unknown functions have played a fundamental role in the development of the theory of differential, integral and integro-differential equations [1–3], and can be used as handy tools in the study of existence, boundedness, uniqueness, stability, invariant manifolds and other qualitative properties of solutions of differential equations and integral equations. For example, the explicit bounds established by the well-known Gronwall–Bellman inequality [4,5], and its nonlinear generalization due to Bihari [6] are used to a considerable extent in the literature. After establishing Gronwall–Bellman inequality, during the past few years, many investigators introduced a huge number of useful inequalities which generalize Gronwall–Bellman inequality in various cases from literature (see [7–31]).

In [3], Pachpatte introduced the following theorem

Theorem 1.1. *Let u, f, g, h and p be nonnegative continuous functions defined on $I = [,\infty)$, and u_0 be a nonnegative constant. If the inequality*

$$u(t) \leqslant u_0 + \int_0^t [f(s)u(s) + p(s)]ds + \int_0^t f(s)\left(\int_0^s g(\sigma)u(\sigma)d\sigma\right)ds, \tag{1.1}$$

holds for $t \in I$, then

$$u(t) \leqslant u_0 + \int_0^t \left[p(s) + f(s)\left\{u_0 \exp\left(\int_0^s [f(\sigma)+g(\sigma)]d\sigma\right) + \int_0^s p(\sigma)\exp\left(\int_\sigma^s [f(\tau)+g(\tau)]d\tau\right)d\sigma\right\}\right]ds, \quad \forall t \in I.$$

Agarwal et al. [7] established the following retarded inequality

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$$u(t) \leqslant a(t) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} g_i(t,s) w_i(u(s)) ds, \quad t_0 < t < t_1.$$

Agarwal et al. [8] examined the following retarded inequality

$$\varphi(u(t)) \leqslant a(t) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s)[f_i(s)\varphi_1(u(s)) + g_i(s)\varphi_2(\log(u(s)))] ds.$$

In this article, some retarded nonlinear integral inequalities are discussed, however, the bound presented on such an inequality in (1.1) is not directly applicable in the study of certain nonlinear retarded differential and integral equations. In some situations, it is desirable to investigate some new inequalities of the above type where non-retarded case t in (1.1) is replaced by retarded case $\alpha(t)$ and the linear case $u(t)$ in integral functions in (1.1) is replaced by the nonlinear case $\varphi(u(t))$, and also established a slight generalization of the celebrated Gronwall–Bellman type inequalities which can be used more effectively in the study the qualitative behavior of the solutions of certain classes of nonlinear retarded differential and integral equations. Some applications of some of our results are also introduced to illustrate the benefits of this work.

2. Main results

In this section, we state and prove some new retarded nonlinear integral inequalities of Gronwall–Bellman–Pachpatte type, which can be used in the analysis of various problems in the theory of retarded nonlinear differential equations.

We shall introduce some notations: $\mathbb{R}$ denoted the set of real numbers, $I = [0,\infty)$ is the subset of $\mathbb{R}$, $'$ denotes the derivative. $\mathcal{C}(I,I)$ denotes the set of all continuous functions from I into I and $\mathcal{C}^1(I,I)$ denotes the set of all continuously differentiable functions from I into I.

Theorem 2.1. *Let $u(t), g(t), f(t) \in \mathcal{C}(I,I)$, $\alpha \in \mathcal{C}^1(I,I)$ be nondecreasing with $\alpha(t) \leqslant t$ on I with $\alpha(0) = 0$ and u_0 be a nonnegative constant. If the inequality*

$$u(t) \leqslant u_0 + \int_0^{\alpha(t)} [f(s)u(s) + q(s)] ds + \int_0^{\alpha(t)} f(s)\left(\int_0^s g(s)u(s) ds\right) ds. \tag{2.1}$$

for all $t \in I$. Then

$$u(t) \leqslant u_0 + \int_0^t \left(\alpha'(s)p(\alpha(s)) + \alpha'(s)f(\alpha(s)) \exp\left(\int_0^{\alpha(s)} [f(\tau) + g(\tau)] d\tau\right) \left[u_0 + \int_0^{\alpha(s)} p(\sigma) \exp\left(\int_0^{\sigma} [f(\tau) + g(\tau)] d\tau\right) d\sigma\right]\right), \tag{2.2}$$

for all $t \in I$.

Proof. Define a function $z(t)$ by the right-hand side of (2.1) which is a nonnegative and nondecreasing function on I with $z(0) = u_0$. Then (2.1) is equivalent to

$$u(t) \leqslant z(t), \; u(\alpha(t)) \leqslant z(\alpha(t)) \leqslant z(t), \quad \forall t \in I. \tag{2.3}$$

Differentiating $z(t)$, with respect to t and using (2.3), we get

$$\begin{aligned} z'(t) &= \alpha'(t)[f(\alpha(t))u(\alpha(t)) + p(\alpha(t))] + \alpha'(t)f(\alpha(t)) \int_0^{\alpha(t)} g(\sigma)u(\sigma) d\sigma \\ &\leqslant \alpha'(t)p(\alpha(t)) + \alpha'(t)f(\alpha(t))[z(t) + \int_0^{\alpha(t)} g(\sigma)z(\sigma) d\sigma]. \end{aligned} \tag{2.4}$$

for all $t \in I$. Define a function $v(t)$ by

$$v(t) = z(t) + \int_0^{\alpha(t)} g(\sigma)z(\sigma) d\sigma, \quad \forall t \in I. \tag{2.5}$$

then $v(0) = z(0) = u_0, z'(t) \leqslant \alpha'(t)(p(\alpha(t)) + f(t)v(t))$ from (2.4), and from (2.5) $z(t) \leqslant v(t), z(\alpha(t)) \leqslant v(\alpha(t)) \leqslant v(t)$. Differentiating $v(t)$, with respect to t, we get

$$v'(t) = z'(t) + \alpha'(t)g(\alpha(t))z(\alpha(t)) \leqslant \alpha'(t)p(\alpha(t)) + \alpha'(t)[f(\alpha(t)) + g(\alpha(t))]v(t), \forall t \in I. \tag{2.6}$$

Integrating the inequality (2.6) from 0 to t implies the estimation

$$v(t) \leqslant \exp\left(\int_0^{\alpha(t)} [f(\tau) + g(\tau)] d\tau\right) \left[u_0 + \int_0^{\alpha(t)} p(\sigma) \exp\left(\int_0^{\sigma} [f(\tau) + g(\tau)] d\tau\right) d\sigma\right] \tag{2.7}$$

for all $t \in I$. Using (2.7) in (2.4), we have