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Numerical resolution of a reinforced random walk model arising in haptotaxis

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ABSTRACT

In this paper we study the numerical resolution of a reinforced random walk model arising in haptotaxis and the stabilization of solutions. The model consists of a system of two differential equations, one parabolic equation with a second order non-linear term (haptotaxis term) coupled to an ODE in a bounded two dimensional domain. We assume radial symmetry of the solutions. The scheme of resolution is based on the application of the characteristics method together with a finite element one. We present some numerical simulations which illustrate some features of the numerical stabilization of solutions.

1. Introduction

A characteristic feature of living organisms is that they respond to the environment in search of food and a reproductive mate, which is called taxis. Corresponding to the type of the external stimulus, various types of taxis are defined, such as haptotaxis, chemotaxis and others.

Chemotaxis is a process whereby living organisms respond to chemical substance by moving toward higher or lower concentrations of the chemical substance, or by aggregating or dispersing. Haptotaxis is closely related to chemotaxis, as it is the directional motility or growth of cells following gradient of cellular adhesion sites or substrate-bound chemoattractants. The gradient of the chemical signal in this case is expressed or bound on a surface, in contrast to the classical model of chemotaxis, in which gradient develops in a soluble fluid. These gradients are naturally present in the extracellular matrix of the body during process such as angiogenesis.

In the majority of the theoretical analysis the signal is transported by diffusion, convection or by some other means. The classical chemotaxis equation was introduced by [11], after [15], as the first model to describe the aggregation of slime mold amoebae due to an attractive chemical substance. The model involves the density distribution of the bacteria u and the chemical concentration v in a coupled system of partial differential equations

$$u_t = \Delta u - \mathrm{div}(u\chi(v)\nabla v),$$

$$v_t - \Delta v = g(u, v),$$

where $u_t = \frac{\partial u}{\partial t}$ and $v_t = \frac{\partial v}{\partial t}$.

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However, in some chemotaxis phenomena, the diffusion of the chemical attractant is ignorable, the walker seems to modify the environment in a strictly local manner and there is little or no transport of the chemical substance.

In an attempt to gain understanding of the mechanisms that causes the aggregation of myxobacteria, which slide over slime trails thereby reinforcing the trails, [14] proposed a model based on reinforced random walks. The system of equations derived by Othmer and Stevens is the following:

$$u_t = \mathrm{div}(D\nabla u - u\chi(v)\nabla v), \tag{1}$$

$$v_t = g(u, v), \tag{2}$$

where D is the diffusion constant and $\chi(v)$ is the chemotactic sensitivity of the bacteria. Both, $\chi(v)$ and $g(u, v)$ depend on the nature of the interaction between the bacteria and the chemical stimulus.

[9] studied a reinforced random walk model in haptotaxis. They considered system 1,2 in a bounded domain $\Omega \subset R^n$ with boundary condition:

$$\left(D\frac{\partial u}{\partial n} - u\chi(v)\frac{\partial v}{\partial n}\right) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{3}$$

where $\frac{\partial u}{\partial n}$ and $\frac{\partial v}{\partial n}$ are outward normal derivatives, the random motility D is assumed to be constant and χ measures the haptotactic sensitivity. The function $g(u, v)$ is assumed to be of the form

$$g(u, v) = \widetilde{h}(u, v)\phi(u, v),$$

where, for some constants $0 \leqslant u_1 \leqslant u_2, v_1 < v_2$, it is satisfied

$$\phi(u, v) > 0, \quad \text{if} \quad u_1 \leqslant u \leqslant u_2, \quad v_1 \leqslant v \leqslant v_2,$$

$$\text{and} \quad \widetilde{h}(u_1, v_1) = \widetilde{h}(u_2, v_2).$$

This problem describes the evolution of a biological species moving along a gradient of the concentration of a second species. Notice that in the case of chemotaxis systems, the second species diffuses in a higher or lower velocity, depending on the process, and it is modelized by a parabolic or elliptic equation.

This kind of equations, containing haptotaxis terms, arise for example in modeling cancer process, as angiogenesis, see for instance [1,12]. These problems also present mathematical challenges whereby several authors have been interested, as the literature shows, see for example [9,13,16], and references therein.

From the mathematical point of view, in [9], it is proved that any stationary state (u^*, v^*) of 1,2 is asymptotically stable provided:

$$g = \widetilde{h}\phi, \quad \phi > 0, \quad \widetilde{h}_p > 0, \quad p\chi\widetilde{h}_p + \widetilde{h}_w < 0 \quad \text{at} \quad (u^*, v^*). \tag{4}$$

If (4) is satisfied, then any solution of 1,2, in a bounded domain with boundary condition (3), and initial values near (u^*, v^*), exists for all $t > 0$ and converges as $t \to \infty$ to a nearby stationary solution $(\overline{u}, \overline{v})$. This assertion means that under assumption (4), solutions tend to an uniform distribution, provided the initial distribution is nearly uniform.

The question about what the behavior of solutions could be when condition (4) is not satisfied was the motivation for the study presented in this paper.

This paper is involved with the numerical resolution of a particular case of system 1,2, considered in [9] as an example. To precise, we shall consider the following parabolic-ODE system posed in a bounded domain $\Omega \subset R^2$,

$$u_t = \mathrm{div}\left(\nabla u - \left(u\frac{\beta}{\alpha + \beta v}\nabla v\right)\right), \quad x \in \Omega, \quad t > 0, \tag{5}$$

$$v_t = u - \mu h(v), \quad x \in \Omega, \quad t > 0, \tag{6}$$

that is, $\chi(v) = \frac{\beta}{\alpha+\beta v}$, with $\mu > 0, \alpha > 0$ and $\beta > 0$, complemented with the boundary condition

$$\left(\frac{\partial u}{\partial n} - u\frac{\beta}{\alpha + \beta v}\frac{\partial v}{\partial n}\right) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{7}$$

and initial data

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega. \tag{8}$$

For this particular case, if assumption

$$\chi(v)h(v) < h'(v) \quad \text{for} \quad u_1 \leqslant u \leqslant u_2, \; v_1 \leqslant v \leqslant v_2, \tag{9}$$

with $v_2 - v_1$ small enough, is verified, then assumption (4) is also satisfied (see [9]).