Applied Mathematics and Computation 256 (2015) 529–539

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A parabolic inverse source problem with a dynamical boundary condition ☆

M. Slodička

Department of Mathematical Analysis, Research Group of Numerical Analysis and Mathematical Modeling (NaM²), Ghent University, Ghent, Belgium

A R T I C L E I N F O

Keywords:
Heat equation
Inverse source problem
Time discretization
Error estimates

A B S T R A C T

An inverse source problem for the heat equation is studied in a bounded domain. A dynamical nonlinear boundary condition (containing the time derivative of a solution) is prescribed on one part of the boundary. This models a non-perfect contact on the boundary. The missing purely time-dependent source is recovered from an additional integral measurement. The global in time existence and uniqueness of a solution in corresponding function spaces is addressed using the backward Euler method for the time discretization. Error estimates for time-discrete approximations are derived.

1. Introduction

Physical problems described by partial differential equations (PDEs) accompanied with non-standard boundary conditions (BCs) have been studied by a number of authors in recent years. The dynamical (evolutionary) BCs are not very common in the mathematical literature. Nevertheless, they appear in many mathematical models including heat transfer [1] in a solid in contact with a moving fluid, thermo-elasticity, diffusion phenomena, problems in fluid dynamics, etc. (see [2–5] and the references therein). This BC is sometimes called Wentzel boundary condition after A.D. Wentzel (see [6,7]). The dynamical boundary condition contains the time derivative of u and this models an imperfect contact with the surrounding area. The inflow depends on the change in time of u at the boundary. A very nice application of such a BC is presented in [8]. It models precipitation of rain into a porous media. If the rainfall rate is fully absorbed by soil then the classical Neumann BC can be applied. But if a region near the boundary becomes saturated, then there is no inflow into the porous media. Thus the inflow rate depends on the ability of soil to absorb. Such a situation can be modelled by a dynamical BC.

Let $\Omega \subset \mathbb{R}^d,\ d \geqslant 1$ be a bounded domain with sufficiently smooth boundary Γ, which is split into two non-overlapping complementary parts Γ_D and Γ_N. We denote the outer normal vector associated with Γ by $\boldsymbol{\nu}$. We consider the following parabolic problem for u accompanied with mixed (Dirichlet and nonlinear evolutionary) BCs

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) = h(t)f(x) & \text{in } \Omega \times (0,T), \\ -\nabla u(x,t) \cdot \boldsymbol{\nu} = \partial_t u(x,t) + \sigma(u(x,t)) & \text{on } \Gamma_N \times (0,T), \\ u(x,t) = 0 & \text{on } \Gamma_D \times (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in \Omega \end{cases} \tag{1}$$

☆ The research was supported by the IAP P7/02-project of the Belgian Science Policy.
E-mail address: marian.slodicka@ugent.be
URL: http://cage.ugent.be/~ms

http://dx.doi.org/10.1016/j.amc.2015.01.103
0096-3003/

with the final time $T > 0$. If all data functions $h,\ f,\ u_0$ and σ are given and they obey appropriate conditions, then the *Direct problem* (1) admits a unique solution u. This can be verified by usual techniques for parabolic equations.

The nonlinear function σ can model radiative BC, i.e. $\sigma(s) = s^3|s|$. Please note that the last example has to be linearized for large values of arguments, which is natural in practical applications (we will adopt Lipschitz continuity of σ). If we do not modify the $\sigma(s)$ for large arguments, then we have to involve the monotone behavior of σ and apply the theory of monotone operators [9,10].

This paper is devoted to the identification of a solely time-dependent source $h(t)$ from a given space average of u, namely

$$m(t) = \int_\Omega u(x,t)\mathrm{d}x, \quad t \in [0,T]. \tag{2}$$

A physical motivation of (2) can be found e.g. in [11, p. 378]. The integral overdetermination (2) within the framework of inverse problems for parabolic, hyperbolic and Navier–Stokes equations has been studied e.g. in [11–14] and the references therein. Inverse problems (IPs) are typically ill posed in the sense of Hadamard – see [15]. This means that there is either no solution in a classical sense, or if there is any, then it might not be unique or might not depend continuously on the data. There are two big goals in IPs: (global or local) existence of a solution and its uniqueness. The usual methodology in IPs relies on suitable parametrization of the problem and involving continuous dependence of a parametrized solution on the parameter itself. Then an cost functional capturing the error between parametrized and exact solutions at a given place is constructed and minimized. Lack of convexity of this functional disturbs the uniqueness of a solution. Therefore a suitable (Tikhonov) regularization of the functional is applied to guarantee its convexity, cf. [16–18]. Minimization is based on the theory of monotone operators and numerically solved by adequate approximation techniques, such as the steepest descend, Ritz or Newton or Levenberg–Marquardt method.

The *Inverse Source Problem* (ISP) studied in this paper consists of finding a couple $(u(x,t), h(t))$ obeying (1) and (2). Let us note that this represents just a simplified study case. One can add material coefficients depending on both (time and place) variables to the PDE and BCs. Such an augmented problem will be solvable (adopting reasonable conditions on data functions) by the same technique described below. For ease of explanation we decided to keep the setting so simple as possible but preserving the substantial parts within the study scenario.

Determination of an unknown source is one of hot topics in IPs. If the source exclusively depends on the space variable, one needs an additional space measurement (e.g. solution at the final time), cf. [11,19–27]. For solely time-dependent source a supplementary time-dependent measurement is needed, cf. [11,28–30]. This means that both kinds of ISPs need totally different additional data.

Semigroup approach for determination of the unknown $h(t)$ in a linear heat equation has been used in [28,31] subject to standard BCs. Abstract IPs with Dirichlet BCs and their applications in mathematical physics have been addressed in [11, chapter 6]. Overdetermination based on boundary measurements has been used in [29,30]. The authors in [29] derived an explicit formula for the Fréchet gradient of the cost functional, which has been minimized by conjugate gradient method.

The added value of this paper relies on the global (in time) solvability of the ISP along with a nonlinear evolutionary BC and on the designed numerical scheme for approximations. Novelty counts on reformulating of the ISP into an appropriate direct (non-local) formulation. We propose an attractive variational technique based on two steps. First, we eliminate h from (1) by (2), which turns out to be possible for a sufficiently smooth solution. Then we prove the well-posedness of the problem. The proposed numerical scheme involves the semi-discretization in time by Rothe's method cf. [32,33]. We show the existence of approximations at each time step of the time partitioning in Lemma 3.1 and we derive suitable stability results. The convergence of approximations towards the exact solution is investigated in Theorem 3.1 in suitable function spaces and the error estimates are derived in Theorem 3.2.

Notations. Denote by $(\cdot,\cdot)$ the standard inner product of $L^2(\Omega)$ and $\|\cdot\|$ its induced norm. When working at the boundary Γ we use a similar notation, namely $(\cdot,\cdot)_\Gamma, L^2(\Gamma)$ and $\|\cdot\|_\Gamma$. By $C([0,T],X)$ we denote the set of abstract functions $w : [0,T] \to X$ endowed with the usual norm $\max_{t\in[0,T]}\|\cdot\|_X$ and $L^p((0,T),X)$ is furnished with the norm $\left(\int_0^T \|\cdot\|_X^p \mathrm{d}t\right)^{\frac{1}{p}}$ with $p > 1$, cf. [34]. The symbol X^* stands for the dual space to X. In that follows $C,\ \varepsilon$ and C_ε denote generic positive constants depending only on the data, where ε is a small one and $C_\varepsilon = C(\frac{1}{\varepsilon})$ is a large one.

The additional measurement (2) is represented by an integral over Ω. Integrating/measuring (1) over Ω and taking into account (2) we have

$$m' - \int_\Omega \Delta u = h \int_\Omega f. \tag{MP}$$

Getting h under control means to regulate $\int_\Omega \Delta u$ assuming that $\int_\Omega f \neq 0$. Thus we need a variational framework for strong solutions. Multiplying the PDE from (1) by a test function $-\Delta\varphi$ and integrating in space we get

$$-(\partial_t u, \Delta\varphi) + (\Delta u, \Delta\varphi) = -h(f, \Delta\varphi).$$

The first term can be rewritten by Green's theorem and the BCs as follows