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Approximation by network operators with logistic activation functions

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ABSTRACT

This paper aims to study the construction and multivariate approximation of a class of network operators with logistic sigmoidal functions. First, a class of even and bell-shaped function with support on $\mathbb{R}$ is constructed by using appropriate translation and combination of the logistic function. Then, the constructed function is employed as activation function to construct a kind of so-called Cardaliaguet–Euvrard type network operators. Finally, these network operators are used to approximate bivariate functions in $C_{[-1,1]^2}$, and a Jackson type theorem for the approximation errors is established.

1. Introduction

A feed-forward neural network (FNN) with one hidden layer can be expressed mathematically as

$$\sum_{i=1}^{N} c_i \phi(y_i \cdot x + \theta_i),$$

where output weights $c_i \in \mathbb{R}$, threshold values $\theta_i \in \mathbb{R}$, the dimension of input weights y_i corresponds to that of the input x, and ϕ is called the activation function of the network.

It is well-known that FNN is an universal approximator for continuous or Lebesgue integrable target function defined on a compact set. In connection with such approximation, there are mainly three problems: density, complexity, and algorithm. At present, the density problem has been satisfactorily solved, and many classical results can be found in much article, such as [1–7]. The complexity problem is to determine how many neurons are necessary to yield a prescribed degree of approximation, which mainly describes the relationship between the rate of approximation and the number of neurons in hidden layer. In the study of this problem, the network operators usually are constructed for approximating target function, and thus the degrees of approximation are estimated (see [8–13]). This paper continues to study this problem.

We notice that Cardaliaguet and Euvrard [14] introduced an interesting network operator called Cardaliaguet Euvrard operator by using the centered bell-shaped continuous function with compact support, and Anastassiou [15] discussed the approximation error for the above Cardaliaguet–Euvrard operators. On the other hand, as we know, the logistic function defined by

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$$\sigma(x) = \frac{1}{1+e^{-x}} \tag{1}$$

is a typical sigmoidal function, i.e., bounded, and satisfying

$$\lim_{x\to+\infty}\sigma(x) = 1, \quad \lim_{x\to-\infty}\sigma(x) = 0,$$

on $\mathbb{R}$, which is usually used as the activation function in FNN. Especially, more papers, such as [16–23], have studied the construction and approximation of network operators with the sigmoidal function. The main purpose of this paper is to study the construction and multivariate approximation of Cardaliaguet–Euvrard type network operators with the logistic function σ. In fact, we will use appropriate translation and combination of function σ and construct a class of even and bell-shaped function with support $\mathbb{R}$. Then, we employ the constructed function as activation function to construct a kind of so-called Cardaliaguet–Euvrard type network operators. Finally, as approximation tool, the constructed network operators are used to approximate bivariate functions in $C_{[-1,1]^2}$, and the approximation errors are estimated.

It should be pointed out that since some analytical techniques and refined estimates are used in this paper, our main result, Theorem 1 below, is concise compared with Theorem 3.2 of [24].

2. Preliminaries and the theorem

For the function defined by (1), set

$$\phi(x) = \frac{1}{2}(\sigma(x+1)-\sigma(x-1)), \quad \phi_d(x) = \frac{1}{d}\phi\left(\frac{x}{d}\right),\ d>0 \tag{2}$$

and

$$\Phi_d(x_1,x_2) = \phi_d(x_1)\phi_d(x_2).$$

By an elementary calculation (or see [20]), we know that

$$\phi(x) = \frac{1}{2}(e-e^{-1})\frac{e^{-x}}{(1+e^{-x-1})(1+e^{-x+1})} = \frac{e^2-1}{2e^2}\,\frac{1}{(1+e^{x-1})(1+e^{-x-1})}.$$

So ϕ is an even function, and

$$\int_{-\infty}^{+\infty}\phi(x)dx = 1, \quad \int_{-\infty}^{+\infty}\phi_d(x)dx = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\phi_d(x_1)\phi_d(x_2)dx_1dx_2 = 1.$$

For $f\in C_{[-1,1]^2}$, we construct operator

$$(F_{n,d}f)(x_1,x_2) = \sum_{k_1=\lceil -n-n^\alpha\rceil}^{[n+n^\alpha]}\sum_{k_2=\lceil -n-n^\alpha\rceil}^{[n+n^\alpha]}\frac{f\left(\frac{k_1}{n+n^\alpha},\frac{k_2}{n+n^\alpha}\right)}{n^{2\alpha}}\Phi_d\left(\frac{nx_1-k_1}{n^\alpha},\frac{nx_2-k_2}{n^\alpha}\right),$$

where $0<\alpha<1$, and $[\cdot], \lceil\cdot\rceil$ are the integral part and ceiling functions, respectively.

Now we give the estimate of approximation errors by the operators $(F_{n,d}f)$ in $C_{[-1,1]^2}$.

Theorem 1. *Let $f\in C_{[-1,1]^2}$, and n, d, and α in operators $(F_{n,d}f)$ satisfy $n^\alpha>2$ and $(dn^\alpha)^{-1}<1$. Then the estimate of approximation errors*

$$|(F_{n,d}f)(x_1,x_2)-f(x_1,x_2)| \leqslant 25\omega\left(f;\frac{2}{n^{1-\alpha}},\frac{2}{n^{1-\alpha}}\right) + \left(\frac{156}{dn^\alpha}+66e^{-\frac{1}{2d}}\right)\|f\|_\infty$$

holds for any $(x_1,x_2)\in[-1,1]^2$, where $\omega(f;\cdot,\cdot)$ denotes the modulus of continuity of f (see [25]), and $\|f\|_\infty$ is the usual maximum norm of f.

To prove Theorem 1, we need some lemmas.

Lemma 1. *For $0<\alpha<1$, we have*

$$\sum_{k=-\infty}^{\infty}\frac{1}{n^\alpha}\phi_d\left(\frac{nx-k}{n^\alpha}\right) \leqslant 4+\frac{1}{dn^\alpha}.$$

Proof. By standard calculation we obtain