Applied Mathematics and Computation 256 (2015) 602–629

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A Godunov-type scheme for the isentropic model of a fluid flow in a nozzle with variable cross-section

Dao Huy Cuong [a,b], Mai Duc Thanh [c,*]

[a] Nguyen Huu Cau High School, 07 Nguyen Anh Thu, Trung Chanh Ward, Hoc Mon District, Ho Chi Minh City, Viet Nam
[b] Department of Mathematics and Computer Science, University of Science, Vietnam National University-Ho Chi Minh City, 227 Nguyen Van Cu str., District 5, Ho Chi Minh City, Viet Nam
[c] Department of Mathematics, International University, Vietnam National University-Ho Chi Minh City, Quarter 6, Linh Trung Ward, Thu Duc District, Ho Chi Minh City, Viet Nam

ARTICLE INFO

Keywords:
Fluid in a nozzle
Cross-section
Hyperbolic conservation law
Nonconservative
Riemann solver
Godunov-type scheme

ABSTRACT

We present a Godunov-type scheme for the isentropic model of a fluid flow in a nozzle with variable cross-section. The model is of nonconservative form, making it hard for standard numerical discretizations of the nonconservative term. In particular, the error for a standard numerical scheme with a usual discretization of the nonconservative term may become larger as the mesh size gets smaller. We first re-investigate the Riemann problem of the model, pointing out several interesting properties of the wave curves, and establishing specific existence domain for each type of solutions. Then, we incorporate local Riemann solutions to build a Godunov-type scheme for the model. The scheme is constructed in subsonic and supersonic regions, where the system is strictly hyperbolic. Tests show that our scheme can capture standing waves, so that it is well-balanced. Furthermore, tests also show that our Godunov-type scheme can give a good accuracy for numerical approximations of exact solutions. Our Godunov-type scheme can resolve the difficulty of other existing schemes for similar models of fluid flows with nonconservative terms.

1. Introduction

In this paper we study to build a Godunov-type scheme for the numerical approximation of weak solutions to the initial-value problem associated with the following isentropic model of a fluid flow in a nozzle with variable cross-section

$$
\begin{aligned}
&\partial_t(a\rho) + \partial_x(a\rho u) = 0,\\
&\partial_t(a\rho u) + \partial_x(a(\rho u^2 + p)) = p\partial_x a, \quad x \in \mathbb{R}, \quad t > 0,
\end{aligned}
\tag{1.1}
$$

where $\rho(x,t), u(x,t), p(x,t)$ denote the density, particle velocity, and pressure of the fluid, respectively, and $a = a(x)$ denotes the cross-section of the nozzle. The two equations in (1.1) represent the balance of mass and momentum, respectively.

Even for a smooth cross-section $a = a(x)$, numerical discretizations will yield piece-wise constant approximate functions. Therefore, the term $p\partial_x a$ is a nonconservative term, and so the system (1.1) is of nonconservative form, see [9]. Numerical approximations for systems of balance laws with nonconservative terms have been a very interesting, but rather challenging topic for many authors. This is because the standard numerical schemes for systems of conservation laws with usual

* Corresponding author.
E-mail addresses: cuongnhc82@gmail.com (D.H. Cuong), mdthanh@hcmiu.edu.vn (M.D. Thanh).

http://dx.doi.org/10.1016/j.amc.2015.01.024
0096-3003/

discretizations of the nonconservative terms often lead to unsatisfactory results. In particular, the error may become larger as the mesh sizes get smaller. Furthermore, oscillations can be seen for this kind of numerical discretizations of the system.

Motivated by the work of LeFloch-Thanh [22] on a Godunov-type scheme for the shallow water equations with variable topography, we aim to build in this paper a Godunov-type scheme for the model of an isentropic fluid in a nozzle with variable cross-section (1.1). The most interesting result of this paper is that our Godunov-type scheme can resolve the difficulty of the well-balanced scheme in [19] when dealing with the resonant cases. Moreover, the scheme can provide better convergence results than the ones in [22] for shallow water equations with variable topography. Observe that a Godunov-type scheme is based on solutions of the local Riemann problem for (1.1), where, as is well-known, we supplement the system (1.1) the trivial equation

$$\partial_t a = 0. \tag{1.2}$$

Note that the system (1.1) and (1.2) can be written in the form of a nonconservative system of balance laws

$$\partial_t U + A(U)\partial_x U = 0,$$

for $U = (\rho, u, a)$, for example, and $A(U)$ is a matrix determined below.

In this work, our review on the Riemann problem for the system (1.1) and (1.2) will give us interesting properties on the wave curves together with characterizations of the existence domain of each kind of Riemann solutions when the initial data belong to different subsonic or supersonic regions. Then, we employ these Riemann solutions to build up a Godunov-type scheme for the model. Tests show that our Godunov-type scheme is well-balanced as it can capture steady solutions. Furthermore, tests also indicate that this Godunov-type scheme possesses a good accuracy.

This paper continues the study on nonconservative systems of balance laws we have pursued for many years, see, for example, [20,21,28,29] for the very related topic on the Riemann problem, and [19,22,30–32] for numerical approximations. The reader is referred to [23,20] for the Riemann problem for the isentropic model (1.1), to [29] for the Riemann problem for the model of a general fluid in a nozzle with discontinuous cross-section, to [21,22,7,25] for the Riemann problem for the shallow water equations with discontinuous topography, to [28,26] for the Riemann problem for two-phase flow models, and to [16,17,12] for the Riemann problem for other hyperbolic nonconservative models. See [10] for the standard Godunov scheme of systems of conservation laws. We note that Godunov-type schemes for various hyperbolic systems of balance laws in nonconservative form were studied in [17,8,22,2,27,26]. Numerical schemes for multi-phase flow models were presented in [1,15,24,31–33]. Well-balanced schemes for a single conservation law with a source term were studied in [14,5,6,11,3]. Numerical schemes for other hyperbolic models in nonconservative form were presented in [4,19,18,13,30]. See also the references therein.

The organization of this paper is as follows. In Section 2 we discuss basic concepts and properties of the system (1.1) and (1.2). Section 3 is devoted to the revisited Riemann problem. In Section 4 we build a Godunov-type scheme for the model (1.1). Section 5 is devoted to numerical tests. Finally, we provide in Section 6 several conclusions and discussions.

2. Preliminaries

2.1. Nonstrict hyperbolicity

For simplicity, in the following we assume that the pressure is given by an equation of state of an isentropic ideal gas

$$p = p(\rho) = \kappa\rho^{\gamma},$$

where $\kappa > 0, 1 < \gamma < 5/3$ are constant. Set

$$h(\rho) = \frac{\kappa\gamma}{\gamma - 1}\rho^{\gamma-1}.$$

Observe that the function h satisfies

$$h'(\rho) = p'(\rho)/\rho.$$

The system (1.1) and (1.2) for any smooth solution $U = (\rho, u, a)^T$ can be re-written in the nonconservative form as

$$\partial_t U + A(U)\partial_x U = 0, \tag{2.1}$$

where

$$A(U) = \begin{pmatrix} u & \rho & \rho u/a \\ h'(\rho) & u & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix $A(U)$ admits the following three eigenvalues

$$\lambda_1(U) = u - \sqrt{p'(\rho)}, \quad \lambda_2(U) = u + \sqrt{p'(\rho)}, \quad \lambda_3(U) = 0. \tag{2.2}$$