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On parameter derivatives of a family of polynomials in two variables

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ABSTRACT

The purpose of the present paper is to give the parameter derivative representations of the form

$$\frac{\partial P_{n,k}(\lambda;x,y)}{\partial \lambda}=\sum_{m=0}^{n-1}\sum_{j=0}^{m} d_{n,j,m}P_{m,j}(\lambda;x,y)+\sum_{j=0}^{k} e_{n,j,k}P_{n,j}(\lambda;x,y)$$

for a family of orthogonal polynomials of variables x and y, with λ being a parameter and $0 \leqslant k \leqslant n; n,k = 0,1,2,\ldots$. First, we shall present the representations of the parameter derivatives of the generalized Gegenbauer polynomials $C_n^{(\lambda,\mu)}(x)$ with the help of the parameter derivatives of the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, i.e. $\frac{\partial}{\partial\alpha}P_n^{(\alpha,\beta)}(x)$ and $\frac{\partial}{\partial\beta}P_n^{(\alpha,\beta)}(x)$. Then, by using these derivatives, we investigate the parameter derivatives for two-variable analogues of the generalized Gegenbauer polynomials. Furthermore, we discuss orthogonality properties of the parametric derivatives of these polynomials.

1. Introduction

In the recent works [1,4–7,9–17], the parameter derivatives of orthogonal polynomials and various special functions which are important in applied mathematics, mathematical and theoretical physics and theoretical engineering have been investigated. In [11,12], Szmytkowski has studied the derivative of the Legendre function of the first kind, with respect to its degree ν, $[\partial P_\nu(z)/\partial\nu]_{\nu=n} (n \in \mathbb{N})$, and its some representations, which are seen in some engineering and physical problems such as in the general theory of relativity and in solving some boundary value problems of potential theory, of electromagnetism and of heat conduction in solids. In [13], explicit expressions of second-order derivative $[\partial^2 P_\nu(z)/\partial\nu^2]_{\nu=0}$ and of third-order derivative $[\partial^3 P_\nu(z)/\partial\nu^3]_{\nu=0}$ have been derived. In [14], the same author has studied the derivative of the associated Legendre function of the first kind of integer degree with respect to its order $\mu, \partial P_n^{\mu}(z)/\partial\mu$. In [15], several representations of the derivative of $P_\nu^{\pm m}(z)$ with respect to the degree ν, for $m \in \mathbb{N}$, which are used in solving boundary value problems of applied mathematics, have been investigated. A relationship between derivatives of the associated Legendre function of the first kind with respect to its order and its degree has been given in [16]. Such derivatives of the associated Legendre function are met in solutions of various problems of theoretical acoustics, heat conduction and other branches of theoretical physics. Furthermore, the papers in [1,4–7,9,10,17] have generally contributed to the theory and applications of orthogonal polynomials. The representation of parametric derivative in the form

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$$\frac{\partial P_n(\lambda;x)}{\partial\lambda}=\sum_{k=0}^{n}c_{n,k}(\lambda)P_k(\lambda;x) \tag{1}$$

for orthogonal polynomials of variable x, with λ being a parameter, has been studied in many papers. Such representations of parametric derivatives have been investigated by Wulkow [17] for discrete Laguerre polynomials, by Froehlich [4] for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, by Koepf [5] for generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ and Gegenbauer polynomials $C_n^{(\lambda)}(x)$, by Koepf and Schmersau [6] for all the continuous and discrete classical orthogonal polynomials. In [10], Szmytkowski has given a method which is different from the methods presented by Froehlich [4] and Koepf [5], and has obtained again the expansions in the form of (1) for Jacobi polynomials, Gegenbauer polynomials and the generalized Laguerre polynomials. The parameter derivative of the generalized Laguerre polynomials is used in relativistic quantum mechanics to determine corrections to wave functions of particles bound in the Coulomb potential. In [9], Ronveaux et al. have presented the recurrence relations for coefficients in the expansion which is more general than the expansion form of (1) as follows

$$\frac{\partial^m P_n(\lambda;x)}{\partial\lambda^m}=\sum_{k=0}^{n}a_{n,k}(m,\lambda)P_k(\lambda;x)\quad (m\in\mathbb{N}).$$

In the paper [7], an alternative approach has been given to obtain iteratively explicit parameter derivative representations of order $m=1,2,\ldots$ for almost all the classical orthogonal polynomial families, i.e., "continuous" classical orthogonal polynomials, classical orthogonal polynomials of a discrete variable or q-classical orthogonal polynomials of the Hahn's class.

The representations of parametric derivatives obtained by Froehlich [4] for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ for $\alpha,\beta>-1$ are as follows

$$\frac{\partial P_n^{(\alpha,\beta)}(x)}{\partial\alpha}=\sum_{k=0}^{n-1}\frac{1}{n+k+\alpha+\beta+1}P_n^{(\alpha,\beta)}(x)+\frac{(\beta+1)_n}{(\alpha+\beta+1)_n}\sum_{k=0}^{n-1}\frac{(2k+\alpha+\beta+1)(\alpha+\beta+1)_k}{(n-k)(n+k+\alpha+\beta+1)(\beta+1)_k}P_k^{(\alpha,\beta)}(x) \tag{2}$$

and

$$\frac{\partial P_n^{(\alpha,\beta)}(x)}{\partial\beta}=\sum_{k=0}^{n-1}\frac{1}{n+k+\alpha+\beta+1}P_n^{(\alpha,\beta)}(x)+\frac{(\alpha+1)_n}{(\alpha+\beta+1)_n}\sum_{k=0}^{n-1}\frac{(-1)^{n+k}(2k+\alpha+\beta+1)(\alpha+\beta+1)_k}{(n-k)(n+k+\alpha+\beta+1)(\alpha+1)_k}P_k^{(\alpha,\beta)}(x), \tag{3}$$

where the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are defined through the Rodrigues formula

$$P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\}$$

and they are orthogonal with respect to the weight function $W_{\alpha,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta},\alpha,\beta>-1$ over the interval $(-1,1)$ (see [8]).

We recall that the Pochhammer symbol is defined by

$$(\alpha)_0=1,\quad (\alpha)_k=\alpha(\alpha+1)\ldots(\alpha+k-1),k=1,2,\ldots.$$

Note that

$$\frac{1}{(\alpha)_n}\frac{\partial}{\partial\alpha}(\alpha)_n=\frac{\partial}{\partial\alpha}(\ln(\alpha)_n)=\frac{\partial}{\partial\alpha}\left(\ln\prod_{k=0}^{n-1}(\alpha+k)\right)=\frac{\partial}{\partial\alpha}\left(\sum_{k=0}^{n-1}\ln(\alpha+k)\right)=\sum_{k=0}^{n-1}\frac{1}{\alpha+k} \tag{4}$$

holds (see [5]).

With motivation from the expansion (1) for orthogonal polynomials in one variable, we consider similar expansion in the form of

$$\frac{\partial P_{n,k}(\lambda;x,y)}{\partial\lambda}=\sum_{m=0}^{n-1}\sum_{j=0}^{m}d_{n,j,m}P_{m,j}(\lambda;x,y)+\sum_{j=0}^{k}e_{n,j,k}P_{n,j}(\lambda;x,y) \tag{5}$$

for orthogonal polynomials of variables x and y, with λ being a parameter and $0\leqslant k\leqslant n;\quad n,k=0,1,2,\ldots$. In a recent paper [1], parametric derivative representations in the form of (5) of Jacobi polynomials on the triangle have been studied. The present paper is devoted to obtain parametric derivatives in the form of (5) for a family of orthogonal polynomials with two variables on a unit disc.

The set up of this paper is summarized as follows. Section 2 contains expansions in the form of (1) for the generalized Gegenbauer polynomials $C_n^{(\lambda,\mu)}(x)$ by using the parameter derivatives of the classical Jacobi polynomials given by (2) and (3). Also, their orthogonality properties are investigated. In Section 3, we obtain representations in the form of (5) for a family of orthogonal polynomials with two variables by means of the relations presented in the previous section. Furthermore, some orthogonality relations for the derivatives of these polynomials are studied.