Applied Mathematics and Computation 256 (2015) 860–875

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Minimum n-rank approximation via iterative hard thresholding

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ARTICLE INFO

Keywords:
Iterative hard thresholding
Low-n-rank tensor recovery
Tensor completion
Compressed sensing

ABSTRACT

The problem of recovering a low n-rank tensor is an extension of sparse recovery problem from the low dimensional space (matrix space) to the high dimensional space (tensor space) and has many applications in computer vision and graphics such as image inpainting and video inpainting. In this paper, we consider a new tensor recovery model, named as minimum n-rank approximation (MnRA), and propose an appropriate iterative hard thresholding algorithm with giving the upper bound of the n-rank in advance. The convergence analysis of the proposed algorithm is also presented. Particularly, we show that for the noiseless case, the linear convergence with rate $\frac{1}{2}$ can be obtained for the proposed algorithm under proper conditions. Additionally, combining an effective heuristic for determining n-rank, we can also apply the proposed algorithm to solve MnRA when n-rank is unknown in advance. Some preliminary numerical results on randomly generated and real low n-rank tensor completion problems are reported, which show the efficiency of the proposed algorithms.

1. Introduction

The problem of recovering an unknown low-rank matrix $X^* \in \mathbb{R}^{m\times n}$ from the linear constraint $\mathcal{A}(X^*) = \mathbf{b}$, where $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^p$ is the linear transformation and $\mathbf{b} \in \mathbb{R}^p$ is the measurement, has been an active topic of recent research with a range of applications including collaborative filtering (the Netflix problem) [1], multi-task learning [2], system identification [3], and sensor localization [4]. One method to solve this inverse problem is to solve the matrix rank minimization problem:

$$\min_{X\in\mathbb{R}^{m\times n}} \operatorname{rank}(X) \quad \text{s.t.} \quad \mathcal{A}(X) = \mathbf{b}, \tag{1}$$

which becomes a mathematical task of minimizing the rank of X such that it satisfies the linear constraint. With the application of nuclear norm which is the tightest convex approach to the rank function, one can relax the non-convex NP-hard problem (1) to a tractable convex one [5,6]. An alternative model of this inverse problem is the minimum rank approximation problem:

$$\min_{X\in\mathbb{R}^{m\times n}} \|\mathbf{b} - \mathcal{A}(X)\|_2 \quad \text{s.t.} \quad \operatorname{rank}(X) \leqslant r, \tag{2}$$

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http://dx.doi.org/10.1016/j.amc.2015.01.099
0096-3003/

where $r = \text{rank}(X^*)$ is known in advance and X^* is the true data to be reconstructed. The model in (2) has been widely studied in the literature [7–12]. In fact, this formulation can not only work for the exact recovery case ($\mathcal{A}(X^*) = \mathbf{b}$), but also suit for the noisy case ($\mathcal{A}(X^*) + \epsilon = \mathbf{b}$), where ϵ denotes the noise by which the measurements are corrupted. Although the model (2) is based on a priori knowledge of the rank of X^*, an incremental search over r, which increases the complexity of the solution by at most factor r, can be applied when the minimum rank r is unknown. Particularly, if an upper bound on r is available, we can use a bisection search over r since the minimum of (2) is monotonously decreasing in r. Then the factor can reduce to $\log r$. Several effective algorithms for solving (2) have been proposed, such as OPTSPACE [10], Space Evolution and Transfer (SET) [11], Atomic Decomposition for Minimum Rank Approximation (ADMiRA) [9] and the Iterative Hard Thresholding (IHT) introduced in [13]. Among these algorithms, the iterative hard thresholding algorithm is an easy-to-implement and fast method, which also shows the strong performance guarantees available with methods based on convex relaxation.

Recently, many researchers focus on the recovery problem in the high dimensional space, which has many applications in computer vision and graphics such as image inpainting [14] and video inpainting. More specifically, by using the n-rank as a sparsity measure of a tensor (or multidimensional array), this inverse problem can be transformed into the mathematical task of recovering an unknown low n-rank tensor $\mathcal{X}^* \in \mathbb{R}^{n_1\times\cdots\times n_N}$ from its linear measurement $\mathbf{b} = \mathcal{A}(\mathcal{X}^*)$ via a given linear transformation $\mathcal{A} : \mathbb{R}^{n_1\times n_2\times\ldots\times n_N} \to \mathbb{R}^p$ with $p \leqslant \prod_{i=1}^N n_i$. Some related studies can be found in [15–19]. In all these studies, the authors mainly discussed the following tensor recovery model:

$$\min_{\mathcal{X}\in\mathbb{R}^{n_1\times\cdots\times n_N}} \sum_{i=1}^N w_i \text{rank}(\mathcal{X}_{\langle i\rangle}) \quad \text{s.t.} \quad \mathcal{A}(\mathcal{X}) = \mathbf{b}, \tag{3}$$

where $\mathcal{X} \in \mathbb{R}^{n_1\times\cdots\times n_N}$ is the decision variable, $\mathcal{X}_{\langle i\rangle}$ is the mode-i unfolding (the notation will be given in Section 2) of $\mathcal{X}$, w_i's are the weighted parameters which satisfy $0 \leqslant w_i \leqslant 1$ and $\sum_{i=1}^N w_i = 1$. Note that (3) can be regarded as an extension of (1) in the high dimensional space $\mathbb{R}^{n_1\times n_2\times\ldots\times n_N}$ and it is a difficult non-convex problem due to the combination nature of the function $\text{rank}(\cdot)$. In order to solve it, the common method is replacing $\text{rank}(\cdot)$ by its convex envelope to get a convex tractable approximation and developing effective algorithms to solve the convex approximation, including FP-LRTC (fixed point continuation method for low n-rank tensor completion) [19], TENSOR-HC (hard completion) [18], and ADM-TR(E) (alternative direction method algorithm for low-n-rank tensor recovery) [15]. Additionally, [20] investigated the exact recovery conditions for the low n-rank tensor recovery problems via its convex relaxation. And lately, [21] studied the problem of robust low n-rank tensor recovery in a convex optimization framework, drawing upon recent advances in robust principal component analysis and tensor completion.

In this paper, we consider a new alternative recovery model extended from problem (2), which is called as *minimum n-rank approximation* (MnRA):

$$\min_{\mathcal{X}\in\mathbb{R}^{n_1\times\cdots\times n_N}} \|\mathcal{A}(\mathcal{X}) - \mathbf{b}\|_2^2 \quad \text{s.t.} \quad \text{rank}(\mathcal{X}_{\langle i\rangle}) \leqslant r_i \quad \forall i = 1,\ldots,N, \tag{4}$$

where $(r_1, r_2, \ldots, r_N)$ is the n-rank of the true data $\mathcal{X}^*$ to be restored (the notation of n-rank will be given in Section 2). Note that this formulation has not been discussed in tensor space in the literature to our knowledge and it also includes both the noisy case ($\mathcal{A}(\mathcal{X}^*) + \epsilon = \mathbf{b}$) and noiseless case ($\mathcal{A}(\mathcal{X}^*) = \mathbf{b}$). One of its special cases is the *low n-rank tensor completion* (LRTC) problem:

$$\min_{\mathcal{X}\in\mathbb{R}^{n_1\times\cdots\times n_N}} \|\mathcal{X}_\Omega - \mathcal{M}_\Omega\|_F^2 \quad \text{s.t.} \quad \text{rank}(\mathcal{X}_{\langle i\rangle}) \leqslant r_i \quad \forall i = 1,\ldots,N, \tag{5}$$

where $\mathcal{X}$, $\mathcal{M}$ are N-way tensors with identical size in each mode, and $\mathcal{X}_\Omega$ (or $\mathcal{M}_\Omega$) denotes the tensor whose $(i_1, i_2, \ldots, i_N)$th component equal to $x_{i_1 i_2\ldots i_N}$ (or $m_{i_1 i_2\ldots i_N}$) if $(i_1, i_2, \ldots, i_N) \in \Omega$ and zero otherwise. To solve (4), we propose an iterative hard thresholding algorithm, which is easy to implement and very fast. Particularly, we prove that for the noiseless case the iterative sequence generated by the proposed algorithm is globally linearly convergent to the true data $\mathcal{X}^*$ with the rate $\frac{1}{2}$ under some conditions, while for the noisy case the distance between the iterative sequence and the true data $\mathcal{X}^*$ is decreased quickly associated with the noise ϵ. Additionally, combining an effective heuristic for determining n-rank, we can also apply the proposed algorithm to solve MnRA (4) when n-rank of $\mathcal{X}^*$ is unknown in advance. Some preliminary numerical results are reported and demonstrate the efficiency of the proposed algorithms.

The rest of our paper is organized as follows. In Section 2, we first briefly introduce some preliminary knowledge of tensor. Then, we propose an iterative hard thresholding algorithm to solve the minimum n-rank approximation problem in Section 3 and the convergence analysis of the proposed algorithm will be presented in Section 4. In Section 5 and Section 6, we give some implementation details and report some preliminary numerical results for low n-rank tensor completion, respectively. Conclusions are given in the last section.

2. Preliminary knowledge

In this section, we briefly introduce some essential nomenclatures and notations used in this paper; and more details can be found in [22]. Scalars are denoted by lowercase letters, e.g., $a, b, c, \ldots$; vectors by bold lowercase letters, e.g., $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$; and matrices by uppercase letters, e.g., $A, B, C, \ldots$. An N-way tensor is denoted as $\mathcal{X} \in \mathbb{R}^{n_1\times\ldots\times n_N}$, whose elements are denoted as