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# On the limit cycles of planar polynomial system with non-rational first integral via averaging method at any order

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ABSTRACT

In this paper, we consider a class of cubic planar polynomial differential system with non-rational first integral. Using the averaging method at any order, we bound the maximum number of limit cycles which bifurcate from the periodic annulus of the origin when we perturb them inside the class of all polynomial systems of degree $n$.



## 1. Introduction and statement of the main results

In the qualitative theory of real planar polynomial differential system, the bifurcation and distribution of limit cycles have been extensively considered in literature, see for instance [4,10]. Limit cycles can bifurcate through many different methods. One of the main methods is perturbing a system which has a center via Poincaré bifurcation, in such a way that limit cycles bifurcate in the perturbed system from the period annulus of the center for the unperturbed system. There are several papers considering the number of limit cycles which bifurcate from the center with rational first integral, see [9] and the references therein. Generally speaking, if the first integral is rational, the integrands of averaged functions are rational, see for instance (6) of [7]. As far as we know, there are only few papers dealing with the limit cycles which bifurcate from the periodic annulus of a center with non-rational first integral, see for instance [2,11,12,14].

One of the good tools for studying the number of limit cycles is the averaging method [16,17]. Roughly speaking, averaging method gives the qualitative relation between the number of limit cycles for differential system and the number of zeros for the averaged function. There are several papers [1,3,6] where the averaging method is extended to study the number of limit cycles which bifurcate from the unperturbed system with an invariant manifold of periodic solutions. In the paper [1], the authors obtain the first order averaged function. If this function vanishes, then the number of limit cycles of perturbed systems depends on the second order averaged function. The authors of the paper [3] consider the second order averaged function. In a recent paper [6], the authors deduce the expression of the averaged function at any order. It is noteworthy that averaging method provides not all the limit cycles, but only provides the limit cycles which generated via Poincaré bifurcation (e.g., [7]) or Hopf bifurcation (e.g., [13]). Moreover, the authors [5] apply averaging method to study the center-focus problem for some analytical planar differential systems.

In the present paper, we consider the following planar polynomial differential system

$$\begin{aligned} \frac{dx}{dt} &= -y(3x^2+y^2)+\varepsilon f(x,y), \\ \frac{dy}{dt} &= x(x^2-y^2)+\varepsilon g(x,y), \end{aligned} \qquad (1)$$

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where $f(x,y)$ and $g(x,y)$ are real polynomials of degree $n$ in the variables $x$ and $y$. Note that the unperturbed system $(1)_{\varepsilon=0}$ has a non-rational first integral $H(x,y) = (x^2+y^2)\exp\left(-\frac{2x^2}{x^2+y^2}\right)$ with integral factor $\mu(x,y) = \frac{2}{x^2+y^2}\exp\left(-\frac{2x^2}{x^2+y^2}\right)$. The origin is a global center for the unperturbed system.

System (1) has been studied in the papers [2,14]. In the paper [14], up to first order averaging method, the authors prove that there are at most $[(n-1)/2]$ limit cycles bifurcating from the periodic annulus of the origin for system (1), where $[\eta]$ denotes the integer part of real number $\eta$. Moreover, this bound is sharp. Later on, using the second order averaging method, the authors of the paper [2] show that there are at most 2 limit cycles bifurcating from the periodic annulus surrounding the origin of system (1) with $n = 3$.

Motivated by the above two papers, in this paper, using the averaging method at any order given by [6], we bound the number of limit cycles which bifurcate from the periodic annulus surrounding the origin of system (1). Our result is the following one:

**Theorem 1.** *Consider system (1) with $|\varepsilon| > 0$ sufficiently small. Let $H_k(n)$ denote the maximum number of limit cycles which bifurcate from the periodic annulus surrounding the origin of system (1) via k order averaging method, then*

(i) $H_1(n) = \left[\frac{n-1}{2}\right]$;
(ii) $H_k(n) \leqslant kn,\ k \geqslant 2$.

**Remark 2.** Though statement (i) has been already contained in the paper [14], we include it in our theorem for the sake of completeness.

In the paper [2], using the second order averaging method, the authors obtain that $H_2(3) = 2$. However, our results show that $H_2(3) \leqslant 6$. It is obvious that this upper bound is not sharp. In order to obtain the sharp upper bound of the number of zeros for second order averaged function, we should impose that the first order averaged function vanishes identically, see for instance [2,15]. In general, the problem of deducing the sharp upper bound of averaged function at any order is very difficult, see for instance [8,18].

The organization of this paper is as follows. In Section 2, we introduce the averaging method at any order. In Section 3, we prove Theorem 1.

## 2. Averaging method at any order

In this section, we state the averaging method at any order given in [6].Consider the following differential equation

$$\frac{dr}{d\theta} = F_0(\theta,r) + \sum_{k\geqslant 1}\varepsilon^k F_k(\theta,r), \tag{2}$$

where $r \in \mathbb{R},\ \theta \in \mathbb{S}^1$, and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with $\varepsilon_0$ a sufficiently small positive value. The functions $F_k(\theta,r)$ are $2\pi$-periodic in the variable $\theta$.

Let $r_0(\theta,z)$ be a particular solution of the unperturbed system $(2)_{\varepsilon=0}$, satisfying that $r_0(0,z) = z \in \mathcal{I}$ with $\mathcal{I}$ a real open interval. We denote by $r_\varepsilon(\theta,z)$ the solution of (2) with initial condition $r_\varepsilon(0,z) = z \neq 0$. Due to the fact that the differential Eq. (2) is analytic, the solution can be written as

$$r_\varepsilon(\theta,z) = r_0(\theta,z) + \sum_{k\geqslant 1}\varepsilon^k r_k(\theta,z). \tag{3}$$

Let $u = u(\theta,z)$ be the solution of the following variational equation

$$\frac{\partial u}{\partial\theta} = \frac{\partial F_0}{\partial r}(\theta, r_0(\theta,z))u, \tag{4}$$

satisfying the initial condition $u(0,z) = 1$.

The next result provides explicit expressions of the function $r_k(\theta,z)$ for any value of $k$.

**Theorem 3.** *The solution (3) of Eq. (2) satisfies $r_k(\theta,z) = u(\theta,z)u_k(\theta,z)$ with*

$$\begin{aligned} u_1(\theta,z) &= \int_0^\theta \frac{F_1(\varphi,r_0(\varphi,z))}{u(\varphi,z)}d\varphi,\\ u_k(\theta,z) &= \int_0^\theta \Bigg(\frac{F_k(\varphi,r_0(\varphi,z))}{u(\varphi,z)} + \sum_{l=0}^{k-2}\sum_{i=1}^{k-l}\frac{1}{i!}\frac{\partial^i F_{k-l-i}}{\partial r^i}(\varphi, r_0(\varphi,z))\\ &\qquad u(\varphi,z)^{i-1}\sum_{j_1+j_2+\cdots+j_i=l+i} u_{j_1}(\varphi,z)u_{j_2}(\varphi,z)\cdots u_{j_i}(\varphi,z)\Bigg)d\varphi, \end{aligned} \tag{5}$$