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Refinements and similar extensions of Hermite–Hadamard inequality for fractional integrals and their applications

Shiow-Ru Hwang [a], Shu-Ying Yeh [b], Kuei-Lin Tseng [c,*]

[a] China University of Science and Technology, Nankang, Taipei 11522, Taiwan
[b] Department of Statistics and Actuarial Science, Aletheia University, Tamsui, New Taipei City 25103, Taiwan
[c] Department of Applied Mathematics, Aletheia University, Tamsui, New Taipei City 25103, Taiwan

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ABSTRACT

In this paper, we establish some new refinements and similar extensions of Hermite–Hadamard inequality for fractional integrals and present several applications in the Beta function.

1. Introduction

Throughout this paper, let $a < b$ in $\mathbb{R}$.

The inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for all convex functions $f : [a,b] \to \mathbb{R}$, is known in the literature as Hermite–Hadamard inequality [7].

For some results which generalize, improve, and extend the inequality (1.1), refer to [1–6,8–18].

In [4], Dragomir and Agarwal established the following results connected with the second inequality in the inequality (1.1).

Theorem A. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{b-a}{8}(|f'(a)|+|f'(b)|), \tag{1.2}$$

which is the trapezoid inequality provided that $|f'|$ is convex on $[a,b]$.

In [12], Kirmaci and Özdemir established the following results connected with the first inequality in the inequality (1.1).

Theorem B. *Under the assumptions of Theorem A, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leqslant \frac{b-a}{8}(|f'(a)|+|f'(b)|), \tag{1.3}$$

which is the midpoint inequality provided that $|f'|$ is convex on $[a,b]$.

In what follows we recall the following definition [13].

* Corresponding author.
E-mail addresses: hsru@cc.cust.edu.tw (S.-R. Hwang), ginger@mail.au.edu.tw (S.-Y. Yeh), kltseng@mail.au.edu.tw (K.-L. Tseng).

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Definition 1. Let $f \in L_1[a,b]$. The Riemann–Liouville integrals $J_{a^+}^{\alpha}f$ and $J_{b^-}^{\alpha}f$ of order $\alpha > 0$ with $a \geqslant 0$ are defined by

$$J_{a^+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt \quad (x > a)$$

and

$$J_{b^-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt \quad (x < b),$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

In [13], Sarikaya et al. established the following Hermite–Hadamard-type inequalities for fractional integrals:

Theorem C. *Let $f : [a,b] \to \mathbb{R}$ be positive with $0 \leqslant a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b) + J_{b^-}^{\alpha}f(a)\right] \leqslant \frac{f(a)+f(b)}{2} \tag{1.4}$$

for $\alpha > 0$.

Theorem D. *Under the assumptions of Theorem A, we have the following Hermite–Hadamard-type inequality for fractional integrals:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b) + J_{b^-}^{\alpha}f(a)\right]\right| \leqslant \frac{2^{\alpha}-1}{2^{\alpha+1}(\alpha+1)}(b-a)\left(|f'(a)| + |f'(b)|\right) \tag{1.5}$$

for $\alpha > 0$.

In [9], Hwang et al. established the following fractional integral inequality:

Theorem E. *Under the assumptions of Theorem A, we have the following Hermite–Hadamard-type inequality for fractional integrals:*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b) + J_{b^-}^{\alpha}f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \\ &\quad \leqslant \frac{(b-a)}{4(\alpha+1)}\left(\alpha - 1 + \frac{1}{2^{\alpha-1}}\right)\left(|f'(a)| + |f'(b)|\right)\end{aligned} \tag{1.6}$$

for $\alpha > 0$.

Remark 1.

(1) *The assumption $f : [a,b] \to \mathbb{R}$ is positive with $0 \leqslant a < b$ in Theorem C can be weakened as $f : [a,b] \to \mathbb{R}$ witha $< b$.*
(2) *In Theorem D, let $\alpha = 1$. Then Theorem D reduces to Theorem A.*
(3) *In Theorem E, let $\alpha = 1$. Then Theorem E reduces to Theorem B.*

In this paper, we establish some new inequalities which refine Hermite–Hadamard inequality (1.1) and Hermite–Hadamard-type inequality (1.4), and we obtain some similar extensions of Theorems A, B, D, E. Some applications for the Beta function are given.

2. New refinements of Hermite–Hadamard-type inequality for fractional integrals

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function with $a < b$. Then we have the inequality*

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\leqslant \frac{3^{\alpha}-1}{4^{\alpha}}f\left(\frac{a+b}{2}\right) + \frac{4^{\alpha}-3^{\alpha}+1}{2\cdot 4^{\alpha}}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \leqslant \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b) + J_{b^-}^{\alpha}f(a)\right] \\ &\leqslant \frac{3^{\alpha}-1}{2\cdot 4^{\alpha}}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] + \frac{4^{\alpha}-3^{\alpha}+1}{2\cdot 4^{\alpha}}[f(a)+f(b)] \leqslant \frac{f(a)+f(b)}{2}\end{aligned} \tag{2.1}$$

for $\alpha > 0$.