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Positive solutions of an elliptic system modeling a population with two age groups

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ABSTRACT

We study the existence of positive solutions for an elliptic system modeling two subpopulations of the same species competing for resources. Under some suitable assumptions, we find the necessary and sufficient conditions for the existence of positive solutions of the system by using bifurcation techniques. Moreover, an open problem proposed by Bouguima et al. (2008) is answered.

1. Introduction

In this paper, we are concerned with the existence of positive solutions of the elliptic system

$$\begin{cases} -\Delta u = a(x)v - e(x)u - c(x)f_1(u,v), & x \in \Omega, \\ -\Delta v = b(x)u - f(x)v - d(x)f_2(u,v), & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded regular domain of $\mathbb{R}^n$ $(n \geqslant 1)$, that is, $\bar{\Omega}$ is an n-dimensional compact connected smooth submanifold of $\mathbb{R}^n$ with boundary $\partial\Omega$, the coefficients a, b, c, d, e and f are $C^\alpha(\bar{\Omega})$ continuous positive functions for a certain $\alpha \in (0,1)$, and $f_i \in C^\alpha(\mathbb{R}^2_+, \mathbb{R}_+)$ $(\mathbb{R}_+ = [0,\infty))$ for $i = 1, 2$.

This system arises from population dynamics where it models the steady-state solutions of the corresponding nonlinear evolution problem [1], where the functions u and v represent, respectively, the concentrations of the adult and juvenile populations. The function a gives the rate at which juveniles become adults and as adults give birth to juveniles, the function b corresponds to the birth rate of the population, the functions e and f reflect the result of harvesting a portion of the population (fishing effort for marine population), c and d measure the competition between u and v. Both populations are living in the same region Ω, and the boundary conditions in (1.1) may be interpreted as the condition that the populations u and v may not stay on $\partial\Omega$. The Laplacian operator shows the diffusive character of u and v within Ω.

Obviously, system (1.1) can be rewritten as

$$\begin{cases} \mathfrak{L}U - A(x)U = F(U), & x \in \Omega, \\ U = 0, & x \in \partial\Omega, \end{cases}$$

where

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$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathfrak{L} = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix}, \quad A(x) = \begin{pmatrix} -e(x) & a(x) \\ b(x) & -f(x) \end{pmatrix}$$

and F the nonlinear term such that

$$F(U)(x) = \begin{pmatrix} c(x)f_1(u,v) \\ d(x)f_2(u,v) \end{pmatrix}.$$

Let k be a fixed positive constant large enough. Then by Lemma 12 in [4], $\mathfrak{L} - A(x) + kI : Y \to Z$ is bijective, and $(\mathfrak{L} - A(x) + kI)^{-1} : Z \to Z$ is strongly positive and compact, where $Y = C_0^{2,\alpha}(\bar{\Omega}) \times C_0^{2,\alpha}(\bar{\Omega}), Z = C^\alpha(\bar{\Omega}) \times C^\alpha(\bar{\Omega})$ and $C_0^{2,\alpha}(\bar{\Omega}) = \left\{u \in C^{2,\alpha}(\bar{\Omega}) : u(x) = 0,\ x \in \partial\Omega\right\}$. For $q \in L^\infty(\Omega)$, let $\rho_1(q) \leqslant \rho_2(q) \leqslant \cdots$ be the eigenvalues of the problem

$$\begin{cases} -\Delta u + qu = \rho u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases}$$

and $\psi_k(q)$ the corresponding eigenfunctions for $k = 1, 2, \ldots$. We denote by $\underline{q} = \operatorname{ess\,inf} q$ and $\bar{q} = \operatorname{ess\,sup} q$ the essential infimum and supremum of q.

When $f_1(u,v) = u(u+v)$ and $f_2(u,v) = v(u+v)$, system (1.1) has been recently studied by several authors, see [1–5] and the references therein. In [1,2], system (1.1) was discussed as a problem in optimal control. Existence and uniqueness results are given in terms of hypotheses which are appropriate for control problems, that is, the coefficients are required to satisfy certain uniform bounds. Brown and Zhang [3] studied system (1.1) subject to Neumann boundary conditions, and obtained more precise existence results in terms of the principal eigenvalue of the linear problem

$$\begin{cases} \mathfrak{L}U - A(x)U = \mu U, & x \in \Omega, \\ \frac{\partial U}{\partial \boldsymbol{n}} = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\boldsymbol{n}$ is the outward unit normal vector on $\partial\Omega$. More concretely, they showed that (1.2) has a principal eigenvalue $\mu_1(A)$ if $a(x) > 0,\ b(x) > 0$ for $x \in \Omega$, that is, the linear system

$$\begin{cases} \mathfrak{L}U = A(x)U, & x \in \Omega, \\ \frac{\partial U}{\partial \boldsymbol{n}} = 0, & x \in \partial\Omega \end{cases}$$

is cooperative, and then by using lower and upper solutions method, they proved that (1.1), subject to Neumann boundary conditions, has a positive solution if and only if $\mu_1(A) < 0$, see [3, Theorem 3.2]. Very recently, Bouguima et al. [4] considered the system (1.1) with $-\Delta$ replaced by some uniformly elliptic operators. By applying fixed point theory, a necessary and sufficient condition, which is $\lambda_1(\mathfrak{L} - A(x)) < 0$, was also achieved for the existence of positive solutions of (1.1), where $\lambda_1(\mathfrak{L} - A(x))$ is the principal eigenvalue of the linear problem

$$\begin{cases} \mathfrak{L}U - A(x)U = \lambda U, & x \in \Omega, \\ U = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

see [4, Corollary 18]. In addition, Bouguima et al. [4] also obtained the following existence and nonexistence results: (i) If $\underline{ab} > \rho_1(e)\rho_1(f)$, then (1.1) has at least one positive solution; (ii) If $\bar{a}\bar{b} \leqslant \rho_1(e)\rho_1(f)$, then (1.1) has no positive solutions, see Theorems 1 and 19 of [4]. However, the following question is unsolved:

Whether or not the existence of positive solutions is controlled by the condition

$$\frac{\bar{a}\bar{b}}{\rho_1(e)\rho_1(f)} > 1. \tag{1.4}$$

The purpose of this paper is to study the existence of positive solutions of more general system (1.1) and to solve the above question by a counterexample.

We shall make the following assumptions:

(H1) a, b, c, d, e and f are $C^\alpha(\bar{\Omega})$ continuous positive functions.
(H2) $f_i \in C^\alpha(\mathbb{R}_+^2, \mathbb{R}_+)$ $(i = 1, 2)$ satisfy

$$f_1(0, y) = 0, \quad \forall\, y \in \mathbb{R}_+; \quad f_2(x, 0) = 0, \quad \forall\, x \in \mathbb{R}_+, \tag{1.5}$$

$$\lim_{x+y \to 0} \frac{f_1(x,y)}{x} = 0, \quad \lim_{x+y \to 0} \frac{f_2(x,y)}{y} = 0. \tag{1.6}$$

(H3) $\lim_{x+y\to\infty} \frac{f_1(x,y)}{x} = \infty,\ \lim_{x+y\to\infty} \frac{f_2(x,y)}{y} = \infty.$