Applied Mathematics and Computation 249 (2014) 133–141

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Wave packet frames on local fields of positive characteristic

Abdullah [b], Firdous A. Shah [a,*]

[a] Department of Mathematics, University of Kashmir, South Campus, Anantnag 192 101, Jammu and Kashmir, India
[b] Department of Mathematics, ZHDC, University of Delhi, New Delhi 110 002, India

ARTICLE INFO

Keywords:
Frame
Wave packet
Local field
Fourier transform

ABSTRACT

The objective of this paper is to construct wave packet frames on local fields of positive characteristic. A necessary and sufficient condition for the wave packet system $\{D_{\mathfrak{p}^j}T_{u(n)a}E_{u(m)b}\psi\}_{j\in\mathbb{Z},\, m,n\in\mathbb{N}_0}$ to be a frame for $L^2(\mathbb{K})$ is given by means of the Fourier transform.

1. Introduction

The concept of frames in a Hilbert space was originally introduced by Duffin and Schaeffer [10] in the context of non-harmonic Fourier series. In signal processing, this concept has become very useful in analyzing the completeness and stability of linear discrete signal representations. Frames did not seem to generate much interest until the ground-breaking work of Daubechies et al. [9]. They combined the theory of continuous wavelet transforms with the theory of frames to introduce wavelet (affine) frames for $L^2(\mathbb{R})$. Since then the theory of frames began to be more widely investigated, and now it is found to be useful in signal processing, image processing, harmonic analysis, sampling theory, data transmission with erasures, quantum computing and medicine. Recently, more applications of the theory of frames are found in diverse areas including optics, filter banks, signal detection and in the study of Bosev spaces and Banach spaces. We refer [3,11] for an introduction to frame theory and its applications.

An important example about frame is wavelet frame, which is obtained by translating and dilating a finite family of functions. One of the fundamental problems in the study of wavelet frames is to find conditions on the wavelet function and the dilation and translation parameters so that the corresponding wavelet system forms a frame. In her famous book, Daubechies [8] proved the first result on the necessary and sufficient conditions for wavelet frames, and then, Chui and Shi [5] gave an improved result. About 10 years later, Casazza and Christenson [2] proved a stronger version of Daubechies sufficient condition for wavelet frames in $L^2(\mathbb{R})$. Recently, Shi and his co-authors [18,24] obtained the necessary conditions and sufficient conditions of wavelet frames.

Another most important concrete realization of frame is Gabor frame. Gabor systems $\{e^{2\pi imax}g(x-nb)\}_{m,n\in\mathbb{Z}}$ are generated by modulations and translations of a single function $g(x)\in L^2(\mathbb{R})$ and hence, can be viewed as the set of time–frequency shifts of $g(x)$ along the lattice $a\mathbb{Z}\times b\mathbb{Z}$ in $\mathbb{R}^2$. Gabor systems that form frames for $L^2(\mathbb{R})$ have a wide variety of applications. An important problem in practice is therefore to determine conditions for Gabor systems to be frames. Many results in this area, including necessary conditions and sufficient conditions have been established during the last two decades. For example, Daubechies [8] proved the first result on the necessary and sufficient conditions for the Gabor system $\{e^{2\pi imax}g(x-nb)\}_{m,n\in\mathbb{Z}}$ to be a frame for $L^2(\mathbb{R})$, Shi and Chen [25] have established a set of necessary conditions for Gabor frames and showed that these conditions are also sufficient for tight frames, Shah [21] extended these results for the real

* Corresponding author.
E-mail addresses: abd.zhc.du@gmail.com (Abdullah), fashah79@gmail.com (F.A. Shah).

http://dx.doi.org/10.1016/j.amc.2014.09.130
0096-3003/

half-line $\mathbb{R}^+$. Recently, Li and Wu [19] presented two new sufficient conditions for Gabor frame via Fourier transform. The conditions they proposed were stated in terms of the Fourier transforms of the Gabor system's generating functions, and the conditions were better than that of Daubechies.

Third important example is the wave packet frame generated by the combined action of dilation, translation and modulation of a single function $\psi \in L^2(\mathbb{R})$. At the first time, systems of this form were used by Córdoba and Fefferman [6] in the study of some classes of singular integral operators. Lebate et al. [14] adopted the same expression to describe any collections of functions which are obtained by applying the same operations to a finite family of functions in $L^2(\mathbb{R}^d)$. In fact, Gabor systems, wavelet systems and the Fourier transform of wavelet systems are special cases of wave packet systems. Wave packet systems have recently been successfully applied to some problems in harmonic analysis and operator theory [15]. Hernandez et al. [13], examined in detail both the continuous and discrete version of wave packet systems by using a unified approach that the authors have developed in their previous work [12]. They gave classification of wave packet systems to be a Parseval frame. Czaja et al. [7] introduced analogs of the notion of Beurling density to describe completeness properties of wave packet systems via geometric properties of the sets of their parameters. In particular, they showed necessary conditions for the wave packet system to be a Bessel system. In 2008, Christensen and Rahimi [4] considered wave packet systems as special cases of generalized shift-invariant systems and presented a sufficient condition for a wave packet system to form a frame. They also presented certain natural conditions on the parameters in a wave packet system which exclude the frame property. Recently, Shah and Abdullah [22] have established a necessary condition for the wave packet system to be frame for $L^2(\mathbb{R})$ by considering that these systems are special cases of generalized shift-invariant systems whereas the later author has given the general characterization of all tight wave packet frames for $L^2(\mathbb{R})$ and $\mathcal{H}^2(\mathbb{R})$ by imposing some mild conditions on the generator in [1].

A field $\mathbb{K}$ equipped with a topology is called a local field if both the additive and multiplicative groups of $\mathbb{K}$ are locally compact Abelian groups. The local fields are essentially of two types (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). The local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$. Examples of local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. Even though the structures and metrics of local fields of zero and positive characteristics are similar, their wavelet and multiresolution analysis theory are quite different. More details are referred to [20,26].

Although there are many results for wavelet and Gabor frames on the real-line $\mathbb{R}$, the counterparts on local field $\mathbb{K}$ of positive characteristic have been reported recently by Li and Jiang [17]. They have constructed tight wavelet frames on local field $\mathbb{K}$ using the Fourier transform and have established a necessary condition and a set of sufficient conditions for the system $\{\psi_{j,k} =: q^{j/2}(\mathfrak{p}^{-j} \cdot -u(k))\}_{j,k\in\mathbb{N}_0}$ to be a frame for $L^2(\mathbb{K})$. Recently, Shah and Debnath [23] have constructed tight wavelet frames on local fields of positive characteristic using the extension principles. They have also presented a sufficient condition for finite number of functions to generate a tight wavelet frame on local field $\mathbb{K}$. On the other hand, Li and Jiang [16] have established a set of necessary and sufficient conditions for the Gabor system $\{g_{m,n} =: \chi_m(bx)g(x-u(n)a)\}_{m,n\in\mathbb{N}_0}$ to be a frame for the local field $\mathbb{K}$ of positive characteristic via Fourier transforms.

The objective of this paper is to investigate wave packet frames on local field $\mathbb{K}$ of positive characteristic by means of the Fourier transform. We establish a necessary and sufficient condition for the wave packet system $\{D_{\mathfrak{p}^j}T_{u(n)a}E_{u(m)b}\psi\}_{j\in\mathbb{Z},\,m,n\in\mathbb{N}_0}$ to be a frame for $L^2(\mathbb{K})$.

The paper is structured as follows. In Section 2, we give a brief introduction to local fields and Fourier analysis on such a field. In Section 3, we present a necessary condition and sufficient condition for the wave packet system $\{D_{\mathfrak{p}^j}T_{u(n)a}E_{u(m)b}\psi\}_{j\in\mathbb{Z},\,m,n\in\mathbb{N}_0}$ to be a frame for $L^2(\mathbb{K})$.

2. Preliminaries on local fields

Let $\mathbb{K}$ be a field and a topological space. Then $\mathbb{K}$ is called a *local field* if both $\mathbb{K}^+$ and $\mathbb{K}^*$ are locally compact Abelian groups, where $\mathbb{K}^+$ and $\mathbb{K}^*$ denote the additive and multiplicative groups of $\mathbb{K}$, respectively. If $\mathbb{K}$ is any field and is endowed with the discrete topology, then $\mathbb{K}$ is a local field. Further, if $\mathbb{K}$ is connected, then $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. If $\mathbb{K}$ is not connected, then it is totally disconnected. Hence by a local field, we mean a field $\mathbb{K}$ which is locally compact, non-discrete and totally disconnected. The p-adic fields are examples of local fields. In the rest of this paper, we use $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let $\mathbb{K}$ be a fixed local field. Then there is an integer $q = \mathfrak{p}^r$, where $\mathfrak{p}$ is a fixed prime element of $\mathbb{K}$ and r is a positive integer, and a norm $|.|$ on $\mathbb{K}$ such that for all $x \in \mathbb{K}$ we have $|x| \geqslant 0$ and for each $x \in \mathbb{K}\setminus\{0\}$ we get $|x| = q^k$ for some integer k. This norm is non-Archimedean, that is $|x+y| \leqslant \max\{|x|,|y|\}$ for all $x,y \in \mathbb{K}$ and $|x+y| = \max\{|x|,|y|\}$ whenever $|x| \neq |y|$. Let dx be the Haar measure on the locally compact, topological group $(\mathbb{K},+)$. This measure is normalized so that $\int_{\mathcal{D}} dx = 1$, where $\mathcal{D} = \{x \in \mathbb{K} : |x| \leqslant 1\}$ is the *ring of integers* in $\mathbb{K}$. Define $\mathcal{B} = \{x \in \mathbb{K} : |x| < 1\}$. The set $\mathcal{B}$ is called the *prime ideal* in $\mathbb{K}$. The prime ideal in $\mathbb{K}$ is the unique maximal ideal in $\mathcal{D}$ and hence as result $\mathcal{B}$ is both principal and prime. Therefore, for such an ideal $\mathcal{B}$ in $\mathcal{D}$, we have $\mathcal{B} = \langle\mathfrak{p}\rangle = \mathfrak{p}\mathcal{D}$.

Let $\mathcal{D}^* = \mathcal{D}\setminus\mathcal{B} = \{x \in \mathbb{K} : |x| = 1\}$. Then, it is easy to verify that $\mathcal{D}^*$ is a group of units in $\mathbb{K}^*$ and if $x \neq 0$, then we may write $x = \mathfrak{p}^k x', x' \in \mathcal{D}^*$. Moreover, each $\mathcal{B}^k = \mathfrak{p}^k\mathcal{D} = \{x \in \mathbb{K} : |x| < q^{-k}\}$ is a compact subgroup of $\mathbb{K}^+$ and usually known as the *fractional ideals* of $\mathbb{K}^+$ (see [20]). Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathcal{B}$ in $\mathcal{D}$, then every element $x \in \mathbb{K}$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Let χ be a fixed character on $\mathbb{K}^+$ that is trivial on $\mathcal{D}$ but is non-trivial on $\mathcal{B}^{-1}$. Therefore, χ is constant on cosets of $\mathcal{D}$ so if $y \in \mathcal{B}^k$, then $\chi_y(x) = \chi(yx),\ x \in \mathbb{K}$. Suppose that χ_u is any character on $\mathbb{K}^+$,