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Some equivalent conditions for block two-by-two matrices to be nonsingular

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ABSTRACT

In this paper, we derive some equivalent conditions for block two-by-two matrices to be nonsingular in an elementary way.

1. Introduction

Throughout this paper, $\mathbb{C}^{m\times n}$ (resp., $\mathbb{R}^{m\times n}$) stands for the set of m by n matrices with complex (resp., real) entries; $\mathrm{rank}(A)$ stands for the rank of a matrix $A \in \mathbb{C}^{m\times n}$; $[\,A \quad B\,]$ denotes a row block matrix consisting of A and B. For integers $m \geqslant 1$ and $1 \leqslant k \leqslant m$, let $\chi_{k,m}$ denote the set

$$\{\alpha : \alpha = (\alpha_1, \ldots, \alpha_k), 1 \leqslant \alpha_1 < \cdots < \alpha_k \leqslant m, \text{ where } \alpha_1, \ldots, \alpha_k \text{ are integers}\}.$$

For $A = (a_{ij}) \in \mathbb{C}^{m\times n}$, let $\alpha = (\alpha_1, \ldots, \alpha_s) \in \chi_{s,m}$ and $\beta = (\beta_r, \ldots, \beta_t) \in \chi_{t,n}$, then the symbol $A_{\alpha,\beta}$ denotes the $s\times t$ submatrix of A determined by rows indexed by α and columns indexed by β, especially, write $A_{\alpha,\rho}(\text{resp.}, A_{\rho,\beta}) = A_{\alpha,\beta}$ when $|\beta| = n(\text{resp.}, |\alpha| = m)$. The symbol $A^{-1}_{\alpha,\beta}$ denotes the inverse of $A_{\alpha,\beta}$, when $|\alpha| = |\beta|$ and $A_{\alpha,\beta}$ is nonsingular. And we can construct two special matrices P_α a $s\times m$ matrix with 1 in positions $(1,\alpha_1), \ldots, (s,\alpha_s)$ and 0 elsewhere, and Q_β an $n\times t$ matrix with 1 in positions $(\beta_1, 1), \ldots, (\beta_t, t)$ and 0 elsewhere. It follows that $P_\alpha A = A_{\alpha,\rho}, AQ_\beta = A_{\rho,\beta}$ and $P_\alpha A Q_\beta = A_{\alpha,\beta}$. For any given $\alpha = (\alpha_1, \ldots, \alpha_k) \in \chi_{k,m}$ we denotes

$$\alpha^c = \{1, \ldots, m\} \setminus \alpha.$$

It is obviously that A_{α^c,β^c} is the $(m-s)\times(n-t)$ submatrix obtained from A by deleting rows indexed by α and columns indexed by β, and

$$\begin{bmatrix} P_\alpha \\ P_{\alpha^c} \end{bmatrix} \in \mathbb{C}^{m\times m} \text{ and } \begin{bmatrix} Q_\beta & Q_{\beta^c} \end{bmatrix} \in \mathbb{C}^{n\times n}$$

are nonsingular. Let $\chi_{0,m}$ denote a null set, especially, when $\alpha \in \chi_{0,m}, \alpha^c = \{1, \ldots, m\}$.

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In the literature, the problem of examining the nonsingularity of

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1.1}$$

have been studied, where $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{n\times m}$ and $D \in \mathbb{C}^{n\times n}$. Decker and Keller [7] obtained some necessary and sufficient conditions for guaranteeing the nonsingularity of the matrix M. Especially, the matrix M is nonsingular if and only if its Schur complement

$$S = D - CA^{-1}B$$

is nonsingular, and $\operatorname{rank}(M) = \operatorname{rank}(A) + \operatorname{rank}(S)$, when the matrix A is nonsingular. Benzi et al. [6] derived a necessary and sufficient condition for M to be nonsingular, when $A \in \mathbb{R}^{m\times m}$ is symmetric positive semidefinite, $C = B^T \in \mathbb{R}^{n\times m}$ has full rank, and $D = 0$. Recently, Bai and Bai [1] and Bai [2] obtained some necessary and sufficient conditions for guaranteeing the nonsingularity of the matrix M, respectively. A detailed discussion of the block two-by-two matrix M defined in (1.1) and its applications can be found in [1,3,5,8–11, etc].

In this paper, we will study the nonsingularity of M in an elementary way.

Lemma 1.1. [4] *Let $A \in \mathbb{C}^{m\times n}$, $\operatorname{rank}(A) = r$, $r \geqslant 1$, $\alpha \in \chi_{r,m}$ and $\beta \in \chi_{r,n}$. If $A_{\alpha,\rho}$ is of full row-rank, and $A_{\rho,\beta}$ is of full column-rank, then $A_{\alpha,\beta}$ is nonsingular.*

Note that, there are many ways to choose $\alpha \in \chi_{r,m}$ such that $A_{\alpha,\rho}$ is a full row-rank matrix, for example, Gaussian elimination, QR Factorization, Singular Value Decomposition (SVD), etc. However, numerically, Gaussian elimination may be unreliable. QR decomposition with pivoting may be a more numerically robust than Gaussian elimination. The SVD is computationally feasible and numerically stable, and is more effective. It seems that the SVD is a good choice.

2. Main results

Theorem 2.1. *Assume that the (1,1) block $A \in \mathbb{C}^{m\times m}$ of the block two-by-two matrix $M \in \mathbb{C}^{(m+n)\times(m+n)}$ defined in (1.1) is zero. Then*

$$\operatorname{rank}(M) = \operatorname{rank}(B) + \operatorname{rank}(C) + \operatorname{rank}\left(\widetilde{D_1}\right), \tag{2.1}$$

where

$$\widetilde{D_1} = D_{\varsigma^c,\tau^c} - D_{\varsigma^c,\tau}B^{-1}_{\varsigma_1,\tau}B_{\varsigma_1,\tau^c} - C_{\varsigma^c,\tau_1}C^{-1}_{\varsigma,\tau_1}D_{\varsigma,\tau^c} + C_{\varsigma^c,\tau_1}C^{-1}_{\varsigma,\tau_1}D_{\varsigma,\tau_1}B^{-1}_{\varsigma_1,\tau}B_{\varsigma_1,\tau^c}, \tag{2.2}$$

$\operatorname{rank}(C) = r_1$, $\operatorname{rank}(B) = r_2$, *$\varsigma \in \chi_{r_1,n}$, $\tau_1 \in \chi_{r_1,m}$, $\varsigma_1 \in \chi_{r_2,m}$ and $\tau \in \chi_{r_2,n}$ such that C_{ς,τ_1} and $B_{\varsigma_1,\tau}$ are nonsingular.*

Proof. Denote $\operatorname{rank}(C) = r_1$ and $\operatorname{rank}(B) = r_2$. Let $\varsigma \in \chi_{r_1,n}$, $\tau_1 \in \chi_{r_1,m}$, $\varsigma_1 \in \chi_{r_2,m}$ and $\tau \in \chi_{r_2,n}$ such that C_{ς,τ_1} and $B_{\varsigma_1,\tau}$ are nonsingular. Write

$$P_1 = \begin{bmatrix} P_\varsigma \\ P_{\varsigma^c} \end{bmatrix} \in \mathbb{C}^{n\times n}, \quad Q_1 = \begin{bmatrix} Q_{\tau_1} & Q_{\tau_1^c} \end{bmatrix} \in \mathbb{C}^{m\times m},$$
$$P_2 = \begin{bmatrix} P_{\varsigma_1} \\ P_{\varsigma_1^c} \end{bmatrix} \in \mathbb{C}^{m\times m}, \quad Q_2 = \begin{bmatrix} Q_{\tau} & Q_{\tau^c} \end{bmatrix} \in \mathbb{C}^{n\times n}.$$

Then P_1 , Q_1, P_2 and Q_2 are nonsingular,

$$P_2BQ_2 = \begin{bmatrix} P_{\varsigma_1} \\ P_{\varsigma_1^c} \end{bmatrix} B \begin{bmatrix} Q_{\tau} & Q_{\tau^c} \end{bmatrix} = \begin{bmatrix} B_{\varsigma_1,\tau} & B_{\varsigma_1,\tau^c} \\ B_{\varsigma_1^c,\tau} & B_{\varsigma_1^c,\tau^c} \end{bmatrix},$$
$$P_1CQ_1 = \begin{bmatrix} P_{\varsigma} \\ P_{\varsigma^c} \end{bmatrix} C \begin{bmatrix} Q_{\tau_1} & Q_{\tau_1^c} \end{bmatrix} = \begin{bmatrix} C_{\varsigma,\tau_1} & C_{\varsigma,\tau_1^c} \\ C_{\varsigma^c,\tau_1} & C_{\varsigma^c,\tau_1^c} \end{bmatrix},$$
$$P_1DQ_2 = \begin{bmatrix} P_{\varsigma} \\ P_{\varsigma^c} \end{bmatrix} D \begin{bmatrix} Q_{\tau} & Q_{\tau^c} \end{bmatrix} = \begin{bmatrix} D_{\varsigma,\tau} & D_{\varsigma,\tau^c} \\ D_{\varsigma^c,\tau} & D_{\varsigma^c,\tau^c} \end{bmatrix}$$

and

$$\begin{bmatrix} P_2 & 0 \\ 0 & P_1 \end{bmatrix}\begin{bmatrix} 0 & B \\ C & D \end{bmatrix}\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & B_{\varsigma_1,\tau} & B_{\varsigma_1,\tau^c} \\ 0 & 0 & B_{\varsigma_1^c,\tau} & B_{\varsigma_1^c,\tau^c} \\ C_{\varsigma,\tau_1} & C_{\varsigma,\tau_1^c} & D_{\varsigma,\tau} & D_{\varsigma,\tau^c} \\ C_{\varsigma^c,\tau_1} & C_{\varsigma^c,\tau_1^c} & D_{\varsigma^c,\tau} & D_{\varsigma^c,\tau^c} \end{bmatrix}.$$