Applied Mathematics and Computation 249 (2014) 182–197

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Positive solutions for a system of fractional differential equations with coupled integral boundary conditions

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ARTICLE INFO

Keywords:
Riemann–Liouville fractional differential equation
Integral boundary conditions
Positive solutions

ABSTRACT

We investigate the existence of positive solutions for a system of nonlinear Riemann–Liouville fractional differential equations with coupled integral boundary conditions.

1. Introduction

Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology (such as blood flow phenomena), economics, control theory, signal and image processing, aerodynamics, viscoelasticity, electromagnetics, and so on (see [11,15,28,32–34]). For some recent developments on the topic, see [1–5,9,10,12,18,19,27,29] and the references therein. Integral boundary conditions arise in thermal conduction problems [13], semiconductor problems [26] and hydrodynamic problems [14]. Coupled boundary conditions appear in the study of reaction–diffusion equations and Sturm–Liouville problems [6,7] and have applications in many fields of sciences and engineering (for example the heat equation [16,17,30]) and mathematical biology [8,31].

In this paper, we consider the system of nonlinear fractional differential equations

$$(S)\qquad \begin{cases} D_{0+}^{\alpha}u(t)+\lambda f(t,u(t),v(t))=0, & t\in(0,1), \quad n-1<\alpha\leqslant n,\\ D_{0+}^{\beta}v(t)+\mu g(t,u(t),v(t))=0, & t\in(0,1), \quad m-1<\beta\leqslant m,\end{cases}$$

with the coupled integral boundary conditions

$$(BC)\qquad \begin{cases} u(0)=u'(0)=\ldots=u^{(n-2)}(0)=0, & u(1)=\int_0^1 v(s)dH(s),\\ v(0)=v'(0)=\ldots=v^{(m-2)}(0)=0, & v(1)=\int_0^1 u(s)dK(s),\end{cases}$$

where $n,\ m\in\mathbb{N},\ n,\ m\geqslant 3$, D_{0+}^{α} and D_{0+}^{β} denote the Riemann–Liouville derivatives of orders α and β, respectively, and the integrals from (BC) are Riemann–Stieltjes integrals.

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http://dx.doi.org/10.1016/j.amc.2014.10.028
0096-3003/

We shall give sufficient conditions on λ, μ, f and g such that positive solutions of $(S)-(BC)$ exist. By a positive solution of problem $(S)-(BC)$ we mean a pair of functions $(u,v) \in C([0,1]) \times C([0,1])$ satisfying (S) and (BC) with $u(t) \geqslant 0,\ v(t) \geqslant 0$ for all $t \in [0,1]$ and $(u,v) \neq (0,0)$. The system (S) with $\alpha = \beta, \lambda = \mu$, and the coupled boundary conditions $u^{(i)}(0) = v^{(i)}(0) = 0$ for $i = 0,1,\ldots,n-2,\ u(1) = av(\xi), v(1) = bu(\eta)$ with $\xi,\ \eta \in (0,1)$ and $0 < ab\xi\eta < 1$ has been investigated in [35]. In that paper, the authors proved the existence of multiple positive solutions, where the functions f and g are continuous and semipositone. The system (S) with uncoupled multi-point boundary conditions

$$(\widetilde{BC}) \quad \begin{cases} u(0) = u'(0) = \ldots = u^{(n-2)}(0) = 0, \quad u(1) = \sum_{i=1}^{p} a_i u(\xi_i), \\ v(0) = v'(0) = \ldots = v^{(m-2)}(0) = 0, \quad v(1) = \sum_{i=1}^{q} b_i v(\eta_i), \end{cases}$$

has been studied in [24]. In [25], the system (S) with $\lambda f(t,u,v)$ and $\mu g(t,u,v)$ replaced by $\tilde{f}(t,v)$ and $\tilde{g}(t,u)$, respectively, (denoted by $(\tilde{S})$) and uncoupled integral boundary conditions

$$(\widetilde{\widetilde{BC}}) \begin{cases} u(0) = u'(0) = \ldots = u^{(n-2)}(0) = 0, \quad u(1) = \int_0^1 u(s)dH(s), \\ v(0) = v'(0) = \ldots = v^{(m-2)}(0) = 0, \quad v(1) = \int_0^1 v(s)dK(s), \end{cases}$$

was investigated. Namely, in [25], we proved the existence and multiplicity of positive solutions of problem $(\tilde{S}) - (\widetilde{\widetilde{BC}})$ $(u(t) \geqslant 0,\ v(t) \geqslant 0,\ t \in [0,1],\ \sup_{t\in[0,1]} u(t) > 0,\ \sup_{t\in[0,1]} v(t) > 0)$ where the nonlinearities f and g are nonsingular or singular functions. We also mention the papers [21], where we studied the existence of positive solutions of the system (S) with $\alpha = n, \beta = m$ and multi-point boundary conditions, and [22,23] where we investigated the existence and multiplicity of positive solutions for system $(\tilde{S})$ with $\alpha = n,\ \beta = m$ and the boundary conditions $(\widetilde{BC})$ under various assumptions on functions f and g.

In Section 2, we present the necessary definitions and properties from the fractional calculus theory and some auxiliary results, which investigate a nonlocal boundary value problem for fractional differential equations. In Section 3, we prove several existence theorems for the positive solutions with respect to a cone for our problem $(S)-(BC)$ which are based on the Guo–Krasnosel'skii fixed point theorem. Finally, some examples are given in Section 4 to illustrate our main results.

2. Auxiliary results

We present here the definitions, some lemmas from the theory of fractional calculus and some auxiliary results that will be used to prove our main theorems.

Definition 2.1. The (left-sided) fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$(I_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad t > 0,$$

provided the right-hand side is pointwise defined on $(0,\infty)$, where $\Gamma(\alpha)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt,\ \alpha > 0$.

Definition 2.2. The Riemann–Liouville fractional derivative of order $\alpha \geqslant 0$ for a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$(D_{0+}^{\alpha} f)(t) = \left(\frac{d}{dt}\right)^n (I_{0+}^{n-\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, \quad t > 0,$$

where $n = [[\alpha]] + 1$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

The notation $[[\alpha]]$ stands for the largest integer not greater than α. We also denote the Riemann–Liouville fractional derivative of f by $D_{0+}^{\alpha} f(t)$. If $\alpha = m \in \mathbb{N}$ then $D_{0+}^{m} f(t) = f^{(m)}(t)$ for $t > 0$, and if $\alpha = 0$ then $D_{0+}^{0} f(t) = f(t)$ for $t > 0$.

Lemma 2.1 [28]. *Let $\alpha > 0$ and $n = [[\alpha]] + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$; that is, n is the smallest integer greater than or equal to α. Then, the solutions of the fractional differential equation $D_{0+}^{\alpha} u(t) = 0,\ 0 < t < 1$, are*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}, \quad 0 < t < 1,$$

where $c_1, c_2, \ldots, c_n$ are arbitrary real constants.