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Convergence and stochastic stability analysis of particle swarm optimization variants with generic parameter distributions

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ABSTRACT

In this paper we present the convergence and stochastic stability analysis of a set of PSO variants: those that differ with the classical PSO in the statistical distribution of the three PSO tuning parameters: inertia weight, local and global acceleration. We provide an analytical expression for the upper limit of the second order stability regions (the so called USL curves) of the particle trajectories that can be applied to most of these PSO algorithms. Thus, this work generalizes to this set the result found in the literature for the classical PSO. We apply this analysis to some of these variants. Finally, numerical experiments have been performed that confirm the known fact that the best algorithm performance is provided tuning the PSO parameters close to the USL curve.

1. Introduction

Particle swam optimization algorithm [10] is inspired in the behavior of bird flocks searching food. A swarm of particles explore the space of solutions and modify its velocity and position in each iteration following the expressions:

$$\begin{aligned} \mathbf{v}_i^{k+1} &= \omega \mathbf{v}_i^k + \phi_1^k(\mathbf{g}^k - \mathbf{x}_i^k) + \phi_2^k(\mathbf{l}_i^k - \mathbf{x}_i^k), \\ \mathbf{x}_i^{k+1} &= \mathbf{x}_i^k + \mathbf{v}_i^{k+1}, \quad i = 1, \ldots, n, \end{aligned} \tag{1}$$

being n the number of individuals or particles in the swarm. The number of vector components in the position and velocity of each particle coincides with the dimension D of the cost function to be optimized. The velocity update consists on three terms: the inertia term, that is related to the velocity in the previous iteration damped by the inertia weight constant; the social term or stochastic vector difference with the global best, $\mathbf{g}^k$, that is, the particle of the swarm k; and the cognitive term or stochastic vector difference with the local best, $\mathbf{l}_i^k$, that is, with the lowest cost function value in iteration k. In the first PSO version ω was set to 1 and ϕ_1^k, ϕ_2^k were uniformly distributed in the intervals $[0, a_g]$ and $[0, a_l]$, being a_g and a_l the global and local acceleration constants. This original PSO algorithm, with $\omega = 1$, was stochastic due to the effect of ϕ_1^k and ϕ_2^k. Particle trajectories oscillated around the position

$$\mathbf{o}_i^k = \frac{\phi_1^k \mathbf{g}^k + \phi_2^k \mathbf{l}_i^k}{\phi_1^k + \phi_2^k}, \tag{2}$$

which is their corresponding oscillation center. To control the stability of the trajectories Shi and Eberhart [20] introduced the inertia weight ω that was originally defined as a real number, aimed at damping the particle velocities.

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First order stability analysis served to prove that the stability of the mean trajectories depended on the parameters $(\omega, \overline{\phi})$ with $\overline{\phi} = \frac{a_g + a_l}{2}$ [24,2,23,26].

Stochastic stability analysis included higher order moments and has proved to be a very useful technique to properly understand the particle swarm dynamics and to clarify the PSO convergence properties [1,5–8,17,18]. These studies proved that (ω, a_g, a_l) points with lowest errors were located close to the upper border of the second order stability region where the variance becomes unbounded.

As it has already been pointed out, the PSO algorithm uses a constant ω, and random uniform distributions for ϕ_1 and ϕ_2. But, how the first and second order stability regions of the particle trajectories will be modified if ω becomes also a random variable, and/or ϕ_1 and ϕ_2 follow other statistical distributions different from the uniform? Some cases have already been studied: for instance, Poli [18] analysed the bare-bones PSO [9]. Also, a social variant of PSO ($a_l = 0$) and a version of FIPS [15] were studied by Poli [17].

This paper aims to generalize the stochastic stability analysis of particle swarm trajectories [18] to different statistical distributions of its parameters ω, ϕ_1 and ϕ_2, obtaining an expression of the upper limit of the second order stability regions (that we will call USL curve for short) that is valid for most of these variants. This analysis cannot be applied if the distributions do not have first order moments, as it is the case for Cauchy [3] and Lévy distribution.

Some numerical studies of this kind of PSO variants have been presented. For instance, Eberhart and Shi [4] used a PSO with random inertia weight to track and optimize dynamic systems. The inertia weight was, in this case, uniformly distributed in $[0.5, 1]$. Kennedy [9] proposed the bare bones PSO with no inertia weight and total acceleration following a normal distribution with mean $(g_j + l_{ij})/2$ (with $i = 1, \ldots, n$ and $j = 1, \ldots, D$) and standard deviation $|g_j - l_{ij}|$ for each particle component. Uniform [16,19], normal [19,25], truncated normal [3], normal-absolute [11,13], and double exponential [12] distributions have been proposed in the literature for all or any of the PSO tuning parameters. Also different PSO variants with a oscillation center constructed with different number of attractors have been proposed [9,14,15].

2. The PSO with arbitrary statistical distributions

Instead of having, as in classical PSO, a constant inertia, ω, and random uniform distributions for ϕ_1 and ϕ_2, we consider a general PSO version,

$$\begin{aligned} \mathbf{v}_i^{k+1} &= \omega^k \mathbf{v}_i^k + \phi_1^k(\mathbf{g}^k - \mathbf{x}_i^k) + \phi_2^k(\mathbf{l}_i^k - \mathbf{x}_i^k), \\ \mathbf{x}_i^{k+1} &= \mathbf{x}_i^k + \mathbf{v}_i^{k+1}. \end{aligned} \tag{3}$$

where ω^k, ϕ_1^k and ϕ_2^k, are random and follow any arbitrary statistical distribution. The PSO version stated in (1) is a particular case of (3) with $\omega \to \delta(\omega - \omega_0)$, where ω_0 is the value of inertia weight that has been adopted and δ is the Dirac distribution, and $\phi_1^k \to U(0, a_g)$, $\phi_2^k \to U(0, a_l)$, where U is the uniform distribution.

2.1. The first order stability region

Algorithm (3) corresponds to the following second order stochastic difference equation

$$\mathbf{x}_i^{k+1} = (\omega^k + 1 - \phi^k)\mathbf{x}_i^k - \omega^k \mathbf{x}_i^{k-1} + \phi_1^k \mathbf{g}^k + \phi_2^k \mathbf{l}_i^k, \tag{4}$$

with $\phi^k = \phi_1^k + \phi_2^k$. Calling

$$\begin{aligned} A &= \omega^k + 1 - \phi^k, \\ B &= -\omega^k, \\ C &= \phi_1^k \mathbf{g}^k + \phi_2^k \mathbf{l}_i^k, \end{aligned} \tag{5}$$

then, the mean trajectories fulfill the second order deterministic difference equation

$$E(\mathbf{x}_i^{k+1}) = E(A)E(\mathbf{x}_i^k) + E(B)E(\mathbf{x}_i^{k-1}) + E(C).$$

This equation can be written as the following deterministic dynamical system:

$$\begin{pmatrix} E(\mathbf{x}_i^{k+1}) \\ E(\mathbf{x}_i^k) \end{pmatrix} = A_\mu \begin{pmatrix} E(\mathbf{x}_i^k) \\ E(\mathbf{x}_i^{k-1}) \end{pmatrix} + c_\mu, \tag{6}$$

where

$$A_\mu = \begin{pmatrix} E(A) & E(B) \\ 1 & 0 \end{pmatrix},$$

and

$$c_\mu = \begin{pmatrix} E(C) \\ 0 \end{pmatrix}.$$