Applied Mathematics and Computation 249 (2014) 389–398

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Iterated integrals of polynomials

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ARTICLE INFO

Keywords:
Iterated polynomials
Real polynomials
Hyperbolic polynomials

ABSTRACT

The paper concerns the decompositions of polynomials onto iterated integrals. This is a continuation of our previous paper (Lorenc, submitted for publication), in which the existence of such decomposition for the Faulhaber polynomials is proven. In the current paper we prove the basic theorem (Theorem 2) presenting the necessary and sufficient conditions for the existence of such decomposition. We discuss these conditions in the wider context of theory of the real and complex polynomials. A number of exemplary decompositions onto iterated integrals of the known classical kinds of polynomials are also presented.

1. Introduction

The immediate cause of preparing this paper was the following problem: which additional assumptions are needed so that for the given sequence of real polynomials $\{p_k(x)\}_{k=1}^{\infty}$, such that $\deg p_k = k$ for every $k \in \mathbb{N}$, there could exist the real numbers $\alpha_{i,k}, i = 0, 1, \ldots, k$ for any $k \in \mathbb{N}$, such that

$$p_k^{(k)}(x) \int_{\alpha_{k,k}}^{n} dx_k \int_{\alpha_{k-1,k}}^{x_k} dx_{k-1} \ldots \int_{\alpha_{1,k}}^{x_2} dx_1 = p_k(n) \tag{$*$}$$

for any $n \in \mathbb{N}$?

In paper [16] we proved that problem $(*)$ has a solution for the sequence $\{S_k(x)\}_{k=1}^{\infty}$ of Faulhaber polynomials, where

$$S_m(x) = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k x^{m+1-k}, \quad m \in \mathbb{N},$$

and B_k are the Bernoulli numbers with $B_1 = \frac{1}{2}$ [13,20,24]. We note that

$$\sum_{k=1}^{n} k^m = S_m(n)$$

for every $m, n \in \mathbb{N}$. In paper [16] all the possible sequences $\{\alpha_{i,k}\}_{i=0}^{k}, k \in \mathbb{N}$, of real numbers for which decomposition $(*)$ holds have been also described.

In this paper we reach a conclusion on more basic problem. We prove when, for given polynomial $p \in \mathbb{R}[x], \deg p \geqslant 2$, the numbers $a_1, a_2, \ldots, a_{\deg p} \in \mathbb{R}$ exist so that the following decomposition holds (see Theorem 2):

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http://dx.doi.org/10.1016/j.amc.2014.10.057
0096-3003/

$$p^{(\deg p)}(x)\int_{\alpha_{\deg p}}^{x} dx_{\deg p}\int_{\alpha_{-1+\deg p}}^{x_{\deg p}} dx_{-1+\deg p}\cdots\int_{\alpha_1}^{x_2} dx_1 = p(x). \tag{1}$$

Besides, we discuss this kind of decomposition for the polynomials of known kinds, including the modified Chebyshev polynomials, Hermite and Bell polynomials, Lagrange polynomials, Laguerre and Jacobi polynomials.

In mentioned Theorem 2 we prove that decomposition (1) holds if and only if any polynomials $p^{(k)}, k = 0, 1, \ldots, -1+\deg p$ have at least one real root. In Section 3 these conditions will be discussed in the context of many known facts of the theory of real and complex polynomials.

Note that the iterated integrals appearing on the left side of (1) describe a volume of some polyhedron with triangular faces with one variable coordinate of one of the vertices.

Other issues connected with decomposition (1), or more precisely with problem $(*)$, such like the completeness of sequence of polynomials $\{p_k(x)\}_{k=1}^{\infty}$ are still under our investigations.

2. Partially solutions

2.1. Discussions on elementary cases

We start the discussion with the following theorem, the aim of which is the complete description of polynomials of degrees 2, 3, and 4, for which the integral decomposition of type $(*)$ exists.

Theorem 1. *We have*:

$$2a_2\int_{\alpha_2}^{x} dx_2\int_{\alpha_1}^{x_2} dx_1 = a_2x^2 + a_1x + a_0, \quad a_2 \neq 0 \tag{2}$$

holds if and only if $\alpha_1 = -\frac{a_1}{2a_2}$ and $\alpha_2 = -\frac{a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_2}$ *whenever* $a_1^2 - 4a_0a_2 \geqslant 0$,

$$6a_3\int_{\alpha_3}^{x} dx_3\int_{\alpha_2}^{x_3} dx_2\int_{\alpha_1}^{x_2} dx_1 = a_3x^3 + a_2x^2 + a_1x,\ a_3 \neq 0 \tag{3}$$

holds if and only if

1. $\alpha_1 = -\frac{a_2}{3a_3}$,
2. $\alpha_2 = \alpha_0 \pm \frac{\sqrt{a_2^2 - 3a_1a_3}}{3a_3}$ *if* $a_2^2 - 3a_1a_3 \geqslant 0$,
3. $\alpha_3 \in \left\{0, \frac{3}{2}\alpha_0 \pm \frac{\sqrt{a_2^2 - 4a_1a_3}}{2a_3}\ \textit{whenever}\ a_2^2 - 4a_1a_3 \geqslant 0\right\}$.

In other words, decomposition (3) *holds whenever* $a_2^2 - 4a_1a_3 \geqslant 0$.

We note that the following decomposition holds (for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$):

$$24a_4\int_{\alpha_4}^{x} dx_4\int_{\alpha_3}^{x_4} dx_3\int_{\alpha_2}^{x_3} dx_2\int_{\alpha_1}^{x_2} dx_1 = a_4x^4 + a_3x^3 + a_2x^2 + a_1x$$

where $a_4 \neq 0$ if and only if $3a_3^2 - 8a_2a_4 \geqslant 0$. Furthermore, the decomposition

$$120a_5\int_{\alpha_6}^{x} dx_6\int_{\alpha_5}^{x_6} dx_5\cdots\int_{\alpha_1}^{x_2} dx_1 = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_0,$$

where $a_1 = 0$ and $a_5 \neq 0$, holds for some $\alpha_1, \alpha_2, \ldots, \alpha_6 \in \mathbb{R}$ if and only if $a_4^2 - \frac{5}{2}a_3a_5 \geqslant 0$.

Conditions of similar kinds are fulfilled for polynomials of higher degree (for example, condition

$$(n-1)a_{n-1}^2 - 2na_na_{n-2} \geqslant 0 \tag{4}$$

must be satisfied for every real polynomial $\sum_{k=0}^{n} a_kx^k$ with $a_n \neq 0, n \geqslant 2$).

Proof. All the relations (equalities and inequalities) follow easily from the following general formula

$$n!a_n\int_{\alpha_{n-1}}^{x} dx_{n-1}\int_{\alpha_{n-2}}^{x_{n-1}} dx_{n-2}\cdots\int_{\alpha_0}^{x_1} dx_0 = a_nx^n - na_n\alpha_0x^{n-1} + n(n-1)a_n\left(\alpha_0\alpha_1 - \frac{1}{2}\alpha_1^2\right)x^{n-2}$$
$$+ n(n-1)(n-2)a_n\left(-\frac{1}{6}\alpha_2^3 + \frac{\alpha_0}{2}\alpha_2^2 - \left(\alpha_0\alpha_1 - \frac{1}{2}\alpha_1^2\right)\alpha_2\right)x^{n-3} + \ldots$$

which implies the following system of equations

$$-na_n\alpha_0 = a_{n-1},$$