Applied Mathematics and Computation 249 (2014) 408–418

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On a multidimensional Hilbert-type integral inequality associated to the gamma function ☆

Michael Th. Rassias [a,b,*], Bicheng Yang [c]

[a] Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA
[b] Department of Mathematics, ETH-Zürich, Rämistrasse 101, 8092 Zürich, Switzerland
[c] Department of Mathematics, Guangdong University of Education, Guangzhou, Guangdong 510303, PR China

ARTICLE INFO

Keywords:
Hilbert-type integral inequality
Weight function
Equivalent form
Hilbert-type integral operator
Gamma function

ABSTRACT

In this paper a multidimensional Hilbert-type integral inequality with the best possible constant factor associated to the gamma function is proved. An equivalent form and some reverses are obtained. We also consider the operator expressions and some particular results related to the non-homogeneous kernels and a homogeneous kernel of degree zero.

1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geqslant 0, f \in L^p(\mathbb{R}_+), g \in L^q(\mathbb{R}_+)$,

$$\|f\|_p = \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} > 0, \quad \|g\|_q > 0,$$

then we have the following Hardy–Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \|f\|_p \|g\|_q, \tag{1}$$

with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$.

If $a_m, b_n \geqslant 0, a = \{a_m\}_{m=1}^\infty \in l^p, b = \{b_n\}_{n=1}^\infty \in l^q$,

$$\|a\|_p = \left\{ \sum_{m=1}^\infty a_m^p \right\}^{\frac{1}{p}} > 0, \quad \|b\|_q > 0,$$

then we still have the discrete variant of the inequality (1) with the same best constant factor $\frac{\pi}{\sin(\pi/p)}$, as follows:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q. \tag{2}$$

Inequalities (1) and (2) are important in mathematical analysis and its applications (cf. [1–5,25]).

☆ Foundation item: This work is supported by the National Natural Science Foundation of China (No. 61370186), and 2012 Knowledge Construction Special Foundation Item of Guangdong Institution of Higher Learning College and University (No. 2012KJCX0079).
* Corresponding author at: Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA.
E-mail addresses: michailrassias@math.princeton.edu, michail.rassias@math.ethz.ch (M.Th. Rassias), bcyang@gdei.edu.cn, bcyang818@163.com (B. Yang).

http://dx.doi.org/10.1016/j.amc.2014.10.037
0096-3003/

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [6] provided an extension of (1) for $p=q=2$. In 2009 and 2011, Yang [3,4] proved some extensions of (1) and (2) as follows:

If $\lambda_1, \lambda_2, \lambda \in \mathbb{R}, \lambda_1+\lambda_2=\lambda$, $k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$, with

$$k(\lambda_1)=\int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}dt \in \mathbb{R}_+,$$

$$\phi(x)=x^{p(1-\lambda_1)-1}, \quad \psi(y)=y^{q(1-\lambda_2)-1}, \quad f(x),g(y)\geqslant 0,$$

$$f\in L_{p,\phi}(\mathbb{R}_+)=\left\{f;\|f\|_{p,\phi}:=\left\{\int_0^\infty \phi(x)|f(x)|^p dx\right\}^{\frac{1}{p}}<\infty\right\},$$

$$g\in L_{q,\psi}(\mathbb{R}_+), \quad \|f\|_{p,\phi},\|g\|_{q,\psi}>0,$$

then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x,y)f(x)g(y)dxdy < k(\lambda_1)\|f\|_{p,\phi}\|g\|_{q,\psi}, \tag{3}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x,y)$ is finite and

$$k_\lambda(x,y)x^{\lambda_1-1} \quad (k_\lambda(x,y)y^{\lambda_2-1})$$

is decreasing with respect to $x>0$ $(y>0)$, then for a_m, $b_n \geqslant 0$,

$$a\in l_{p,\phi}=\left\{a;\|a\|_{p,\phi}:=\left\{\sum_{n=1}^\infty \phi(n)|a_n|^p\right\}^{\frac{1}{p}}<\infty\right\},$$

$$b=\{b_n\}_{n=1}^\infty \in l_{q,\psi}, \quad \|a\|_{p,\phi},\|b\|_{q,\psi}>0,$$

we have

$$\sum_{m=1}^\infty\sum_{n=1}^\infty k_\lambda(m,n)a_m b_n < k(\lambda_1)\|a\|_{p,\phi}\|b\|_{q,\psi}, \tag{4}$$

where, the constant factor $k(\lambda_1)$ is still the best possible.

For

$$\lambda=1, \quad k_1(x,y)=\frac{1}{x+y}, \quad \lambda_1=\frac{1}{q}, \quad \lambda_2=\frac{1}{p},$$

it is evident that (3) reduces to (1), while (4) reduces to (2). Some other results including multidimensional Hilbert-type inequalities are studied in [7–24], which have provided some new methods for the study of these kinds of inequalities.

In this paper, by applying methods of weight functions and techniques of real analysis, we prove a multidimensional Hilbert-type integral inequality with the best possible constant factor related to gamma function. An equivalent form and some reverses are also obtained. Additionally, we consider the operator expressions and some particular results related to the non-homogeneous kernels and a homogeneous kernel of degree 0.

2. Some lemmas

If $m,n\in\mathbb{N}$ ($\mathbb{N}$ is the set of positive integers), $\alpha,\beta>0$, we set

$$\|x\|_\alpha := \left(\sum_{k=1}^m |x_k|^\alpha\right)^{\frac{1}{\alpha}} (x=(x_1,\ldots,x_m)\in\mathbb{R}^m),$$

$$\|y\|_\beta := \left(\sum_{k=1}^n |y_k|^\beta\right)^{\frac{1}{\beta}} (y=(y_1,\ldots,y_n)\in\mathbb{R}^n).$$

Lemma 1. *If $s\in\mathbb{N},\gamma,M>0,\Psi(u)$ is a non-negative measurable function in $(0,1]$, and*

$$D_M := \left\{x\in\mathbb{R}_+^s; 0<u=\sum_{i=1}^s\left(\frac{x_i}{M}\right)^\gamma\leqslant 1\right\},$$