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Annulus containing all the zeros of a polynomial

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ABSTRACT

Recently Dalal and Govil (2013) proved that for any sequence of positive numbers $\{A_k\}_{k=1}^n$ such that $\sum_{k=1}^n A_k = 1$, a complex polynomial $P(z) = \sum_{k=0}^n a_k z^k$ with $a_k \neq 0, 1 \leqslant k \leqslant n$ has all its zeros in the annulus $\mathcal{C} = \{z : r_1 \leqslant |z| \leqslant r_2\}$, where

$$r_1 = min_{1\leqslant k\leqslant n}\left\{A_k\left|\frac{a_0}{a_k}\right|\right\}^{1/k} \text{ and } r_2 = max_{1\leqslant k\leqslant n}\left\{\frac{1}{A_k}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}.$$

They also showed that their result includes as special cases, many known results in this direction. In this paper we prove that the bounds obtained by making choice of different $\{A_k\}_{k=1}^n$'s cannot be in general compared, that is one can always construct examples in which one result gives better bound than the other and vice versa. Also, we provide a result which gives better bounds than the existing results in all cases. Finally, using MATLAB, we compare the result obtained by our theorem with the existing ones to show that our theorem gives sharper bounds than many of the results known in this direction.

1. Introduction

The properties of polynomials have been studied since the time of Gauss and Cauchy, and have played an important role in many scientific disciplines (see [11–13]). Problems involving location of their zeros find important applications in many areas of applied mathematics such as control theory, signal processing, communication theory, coding theory, cryptography, combinatorics, and mathematical biology. Since Abel and Ruffini proved that there is no general algebraic solution to polynomial equations of degree five or higher, the problem of finding an annulus containing all the zeros of a polynomial became much more interesting and over a period a large number of results have been provided in this direction. An accurate estimate of the annulus containing all the zeros of a polynomial can considerably reduce the amount of work needed to find exact zeros, and so there is always a need for better and better estimates for the annulus containing all the zeros of a polynomial.

The earliest result concerning the location of the zeros of a polynomial is probably due to Gauss, who showed that a polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0.$$

with all a_k real, has no zeros outside certain circle $|z| = R$, where

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$$R = max_{1\leqslant k\leqslant n}(n2^{1/2}|a_k|)^{1/k}.$$

Cauchy [2] improved the above result of Gauss by proving the following

Theorem 1.1. *Let* $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$*, be a complex polynomial. Then all the zeros of* $p(z)$ *lie in the disc*

$$\{z : |z| < \eta\} \subset \{z : |z| < 1 + A\},$$

where

$$A = max_{0\leqslant k\leqslant n-1}|a_k|,$$

and η *is the unique positive root of the real-coefficient equation*

$$z^n - |a_{n-1}|z^{n-1} - |a_{n-2}|z^{n-2} - \ldots - |a_1|z - |a_0| = 0.$$

The above result of Cauchy has been sharpened, among others, by Joyal, Labelle and Rahman [9], Datt and Govil [4], Sun and Hsieh [14], Jain [8], and Affane-Aji, Biaz and Govil [1].

The following result, which provides an annulus containing all the zeros of a polynomial is due to Diaz-Barrero [5].

Theorem 1.2. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k \, (a_k \neq 0, 1 \leqslant k \leqslant n)$ *be a non-constant complex polynomial. Then all its zeros lie in the annulus* $\mathcal{C} = \{z : r_1 \leqslant |z| \leqslant r_2\}$ *, where*

$$r_1 = \frac{3}{2} min_{1\leqslant k\leqslant n}\left\{\frac{2^n F_k C(n,k)}{F_{4n}}\left|\frac{a_0}{a_k}\right|\right\}^{1/k} \tag{1.1}$$

and

$$r_2 = \frac{2}{3} max_{1\leqslant k\leqslant n}\left\{\frac{F_{4n}}{2^n F_k C(n,k)}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}. \tag{1.2}$$

Here F_k *is the* k^{th} *Fibonacci number, namely,* $F_0 = 0,\ F_1 = 1$ *and for* $k \geqslant 2,\ F_k = F_{k-1} + F_{k-2}$*. Furthermore,* $C(n,k) = \frac{n!}{k!(n-k)!}$ *are the binomial coefficients.*

Another result in this direction providing annulus containing all the zeros of a polynomial is the following, and is due to Kim [10].

Theorem 1.3. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k \, (a_k \neq 0, 1 \leqslant k \leqslant n)$ *be a non constant polynomial with complex coefficients. Then all the zeros of p(z) lie in the annulus* $A = \{z : r_1 \leqslant |z| \leqslant r_2\}$ *where,*

$$r_1 = min_{1\leqslant k\leqslant n}\left\{\frac{C(n,k)}{2^n - 1}\left|\frac{a_0}{a_k}\right|\right\}^{1/k} \tag{1.3}$$

and

$$r_2 = max_{1\leqslant k\leqslant n}\left\{\frac{2^n - 1}{C(n,k)}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}. \tag{1.4}$$

Here $C(n,k)$ *are the binomial coefficients.*

The following two results by Diaz-Barrero and Egozcue [7], also provide annuli containing all the zeros of a polynomial.

Theorem 1.4. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k \, (a_k \neq 0, 1 \leqslant k \leqslant n)$ *be a non-constant complex polynomial. Then for* $j \geqslant 2$*, all its zeros lie in the annulus* $C = \{z : r_1 \leqslant |z| \leqslant r_2\}$ *where,*

$$r_1 = min_{1\leqslant k\leqslant n}\left\{\frac{C(n,k)A_k B_j^k (bB_{j-1})^{n-k}}{A_{jn}}\left|\frac{a_0}{a_k}\right|\right\}^{1/k} \tag{1.5}$$

and

$$r_2 = max_{1\leqslant k\leqslant n}\left\{\frac{A_{jn}}{C(n,k)A_k B_j^k (bB_{j-1})^{n-k}} A_{jn}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}. \tag{1.6}$$

Here, $B_n = \sum_{k=0}^{n-1} r^k s^{n-1-k}$ *and* $A_n = cr^n + ds^n$*, where c, d are real constants and r,s are the roots of the equation* $x^2 - ax - b = 0$ *in which* a, b *are strictly positive real numbers. For* $j \geqslant 2, \sum_{k=0}^{n} C(n,k)(bB_{j-1})^{n-k} B_j^k A_k = A_{jn}$*. Furthermore,* $C(n,k)$ *are the binomial coefficients.*