Applied Mathematics and Computation 249 (2014) 444–452

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# Ternary 2N-point Lagrange subdivision schemes

Shahid S. Siddiqi [a], Kashif Rehan [b,*]

[a] Department of Mathematics, University of the Punjab, Lahore 54590, Pakistan
[b] Department of Mathematics, University of Engineering and Technology, KSK Campus, Lahore, Pakistan

ARTICLE INFO

*Keywords:*
Ternary
Lagrange polynomial
Subdivision scheme
Mask
Curves

ABSTRACT

Ternary 2*N*-point Lagrange subdivision schemes with a tension parameter are analyzed. It is shown that the resulting curves are $C^3$, $C^4$ and $C^5$ continuous for the values of $N = 2, 3$ and 4 respectively. The role of the tension parameter in subdivision scheme is demonstrated by a few examples.



## 1. Introduction

Subdivision schemes [2,3,8,10,15] provide useful techniques for the creation of smooth curves from discrete set of data points. These schemes play a significant role in computer aided geometric design, computer aided design, image processing and animation industries *etc*. The popularity of subdivision schemes is due to fact that subdivision techniques are simple and easy to handle.

Dubuc [5] and independently Dyn et al. [8] introduced a binary 4-point interpolating scheme that generates the limiting curve of $C^1$ continuity. Deslauriers and Dubuc [3] analyze b-ary 2*N*-point schemes derived from polynomial interpolation. For a fix value of the tension parameter, Dyn et al. [8] subdivision scheme exactly behaves like Dubuc [5] scheme. Weissman [16] introduced a binary 6-point interpolating subdivision scheme that generates the limiting curve of $C^2$ continuity. Dyn et al. [7] proposed a binary 4-point approximating subdivision scheme that generates the limiting curve of $C^2$ continuous. Hassan et al. [11] developed a ternary 4-point interpolating subdivision scheme that generates $C^2$ limiting curves for the certain range of tension parameter. Ko et al. [13] presented a ternary 4-point approximating subdivision scheme that generates $C^2$ limiting curve with approximating order 4. Beccari et al. [1] introduced a non-stationary ternary 4-point interpolating subdivision scheme that generates $C^2$ limiting curve with a single tension parameter. Siddiqi et al. [17] calculated the generation of fractal curves and surfaces using ternary 4-point interpolating subdivision scheme [11].

In this paper, a step of subdivision can be taken as a simple and highly local stage. By developing this stage of the subdivision step, we can generates families of limiting curves. Such improvement can lead to improved behaviour of the subdivision scheme. For this, ternary 2*N*-point Lagrange subdivision schemes are presented using tension parameter in $f_{3i}^{k+1}$ which relax the interpolating property to achieve higher smoothness of the limiting curves.

### 1.1. The subdivision scheme

A ternary 4-point subdivision scheme for the design of smooth curve is introduced. Suppose the data points are given as $f_i$, for $i \in Z$. Also set $f_i^0 = f_i$, for $i \in Z$, and define for each $k = 0, 1, 2, \ldots$, and $i \in Z$,

* Corresponding author.
*E-mail addresses:* shahidsiddiqiprof@yahoo.co.uk (S.S. Siddiqi), kkashif_99@yahoo.com (K. Rehan).

http://dx.doi.org/10.1016/j.amc.2014.10.062
0096-3003/

$$f_{3i}^{k+1} = (-\mu)f_{i-2}^k + (4\mu)f_{i-1}^k + (1-6\mu)f_i^k + (4\mu)f_{i+1}^k - (\mu)f_{i+2}^k,$$
$$f_{3i+1}^{k+1} = \frac{-5}{81}f_{i-1}^k + \frac{20}{27}f_i^k + \frac{10}{27}f_{i+1}^k - \frac{4}{81}f_{i+2}^k, \tag{1.1}$$
$$f_{3i+2}^{k+1} = \frac{-4}{81}f_{i-1}^k + \frac{10}{27}f_i^k + \frac{20}{27}f_{i+1}^k - \frac{5}{81}f_{i+2}^k.$$

This scheme comes from interpolating the data $(3^{-k}(i+j), f_{i+j}^k), j = -N+1, 0, \ldots, N$ where $N = 2$, by a cubic polynomial and evaluating it at $3^{-k}(i+1/3)$ and $3^{-k}(i+2/3)$ for the values $f_{3i+1}^k$ and $f_{3i+2}^k$ respectively. It is sufficient to consider $p_3$, the cubic polynomial such that $p_3(j) = f_j$, for $j = -N+1, 0, \ldots, N$, where $N = 2$ for ternary 4-point Lagrange subdivision scheme. Since

$$p_3(t) = \sum_{j=-N+1}^{N} L_j(t) f_j, \quad L_j(t) = \prod_{k=-N+1, k\neq j}^{N} \frac{t-k}{j-k}, \tag{1.2}$$

Now, for the construction of the $f_{3i}^{k+1}$ mask of the subdivision scheme $S$, following Lemma 2.2 of Kim et al. [12] which gives

$$p_3(0) = \sum_{j=-N}^{N} L_j(0) f_j,$$

where

$$L_j(0) = \delta_{j,0} - \mu \prod_{k=-N, k\neq j}^{N} \frac{-N-k}{j-k}.$$

The main purpose behind this idea using tension parameter in mask $f_{3i}^{k+1}$ basically is that it gives flexibility to generate families of approximating curves of $C^1, C^2$ and $C^3$ continuities for ternary 4-point Lagrange subdivision scheme. And it also generates an interpolating curve of $C^1$ continuity for tension parameter $\mu = 0$ which is exactly same as defined ternary 4-point Dubuc and Deslauriers interpolating subdivision scheme in [11]. Beauty of proposed tension parameter effects locally near the control points and helps to improve the performance of the limiting curves with the small increase of support size. Also same way adopted to extend the same idea towards ternary 2$N$-point Lagrange subdivision scheme to achieve maximum degree of smoothness of interpolating and approximating limiting curves.

## 2. Convergence analysis – necessary conditions

Matrix formalism allows us to derive necessary conditions for a subdivision scheme to be $C^m$ based on the eigenvalues of the subdivision matrices. Suppose the eigenvalues are $\lambda_i$, where $\lambda_0 = 1$ and $|\lambda_i| \geqslant |\lambda_{i+1}|$, *for all* $i \in N$. Then following Doo and Sabin [4] terminology, it can be verified that the curvature of the limit function is bounded i.e., $\lambda_1^2 = |\lambda_2| > |\lambda_3|$ which is a necessary condition if the limit function is to have $C^2$ continuity.

Consider the original vertices $\{A, B, C, D, E, F\}$ and the new vertices $\{a, b, c, d, e, f\}$, the subdivision matrix form along the mid point is

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} \frac{-4}{81} & \frac{10}{27} & \frac{20}{27} & \frac{-5}{81} & 0 & 0 \\ -\mu & 4\mu & 1-6\mu & 4\mu & -\mu & 0 \\ 0 & \frac{-5}{81} & \frac{20}{27} & \frac{10}{27} & \frac{-4}{81} & 0 \\ 0 & \frac{-4}{81} & \frac{10}{27} & \frac{20}{27} & \frac{-5}{81} & 0 \\ 0 & -\mu & 4\mu & 1-6\mu & 4\mu & -\mu \\ 0 & 0 & \frac{-5}{81} & \frac{20}{27} & \frac{10}{27} & \frac{-4}{81} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix}.$$

The eigenvalues for this are $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}(-4+243\mu), \frac{1}{81}(-4+405\mu)$.

Similarly, the vertex subdivision matrix of the proposed subdivision scheme is defined as

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} \frac{-5}{81} & \frac{20}{27} & \frac{10}{27} & \frac{-4}{81} & 0 \\ \frac{-4}{81} & \frac{10}{27} & \frac{20}{27} & \frac{-5}{81} & 0 \\ -\mu & 4\mu & 1-6\mu & 4\mu & -\mu \\ 0 & \frac{-5}{81} & \frac{20}{27} & \frac{10}{27} & \frac{-4}{81} \\ 0 & \frac{-4}{81} & \frac{10}{27} & \frac{20}{27} & \frac{-5}{81} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \end{pmatrix}.$$

The eigenvalues for this are $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}(11-486\mu)$ which clearly shows that the curvature of the limit function is bounded.

For the tension parameter $\mu = 0$, we recover the coefficients for the ternary 4-point Dubuc–Deslauriers subdivision scheme. At the vertex, ternary 4-point Dubuc–Deslauriers subdivision scheme has eigenvalues 1, 1/3, 11/81, 1/9, 1/27, where the third eigenvalue is greater than 1/9. So the ternary 4-point Dubuc–Deslauriers subdivision scheme has unbounded curvature.