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A note on high order Bernoulli numbers and polynomials using differential equations

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ABSTRACT

An ordinary differential equation is constructed to determine coefficients of a recurrence formula related to the generating function of Bernoulli numbers. This construction is more complicated work than the case of Eulerian numbers and polynomials. Solving this differential equation, we derive some identities on Bernoulli numbers and polynomials of higher order.

1. Introduction

The generating function of Eulerian polynomial $H_n(x|u)$ is defined by

$$\frac{1-u}{e^t-u}e^{xt}=\sum_{n=0}^{\infty}H_n(x|u)\frac{t^n}{n!}, \tag{1.1}$$

where $u \in \mathbb{C}$ with $u \neq 1$. In the special case, $x=0$, $H_n(0|u)=H_n(u)$ is called the nth Eulerian number (see [1,2,4,7,8]). Sometimes that is called the nth Frobenius–Euler number.

In [5], Kim constructed a nonlinear ordinary differential equation with respect to t which was related to the generating function of Eulerian polynomial. Some identities on Eulerian polynomials of higher order were derived using the differential equation. In [2], Choi considered nonlinear ordinary differential equations with respect to u not t to obtain different identities on Eulerian polynomial.

The generating function of q-Euler polynomial with weight 0 is defined by

$$\frac{2}{qe^t+1}e^{xt}=\sum_{n=0}^{\infty}\widetilde{E}_{n,q}(x)\frac{t^n}{n!}, \tag{1.2}$$

and

$$\widetilde{E}_{0,q}=\frac{2}{1+q}, \quad q\left(\widetilde{E}_q+1\right)^n+\widetilde{E}_{n,q}=0, \quad \text{if } n>0, \tag{1.3}$$

with the usual convention of replacing $\widetilde{E}_q^n$ by $\widetilde{E}_{n,q}$. In the case, $x=0, \widetilde{E}_{n,q}(0)=\widetilde{E}_{n,q}$ is the nth q-Euler number with weight 0 (see [3,6,8]). Here q is a complex number with $|q|<1$. As $q \to 1$, we obtain the well-known definition of Euler polynomials from (1.2) and (1.3).

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In [3], Choi et al. solved the first order partial differential equation to determine coefficients of Nth order ordinary differential equation. The high order ordinary differential equation was used to obtain some identities on q-Euler polynomials of higher order.

As the well-known definition, Bernoulli polynomial $B_n(x)$ is given by

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \tag{1.4}$$

with the usual convention of replacing $B^n(x)$ by $B_n(x)$. In the case, $x=0, B_n(0)=B_n$ is the nth Bernoulli number.

For $N\in\mathbb{N}$, Bernoulli polynomial of order N is defined by the generating function as follows:

$$B^N(t,x)=\underbrace{\left(\frac{t}{e^t-1}\right)\times\cdots\times\left(\frac{t}{e^t-1}\right)}_{N-\text{times}}e^{xt}=\sum_{n=0}^{\infty}B_n^{(N)}(x)\frac{t^n}{n!}. \tag{1.5}$$

In the case, $x=0, B_n^{(N)}(0)=B_n^{(N)}$ is called the nth Bernoulli number of order N.

In Section 2, we construct a nonlinear ordinary differential equation with respect to t. This work is more complicated than the case of Eulerian numbers and polynomials. Because it seems impossible that we obtain a differential equation directly from the recurrence relations of coefficients of kth derivatives of generating function of Bernoulli numbers. To overcome this situation, we introduce new recurrence relations that are sufficient conditions of the original recurrence relations. From this recurrence relations, we obtain an ordinary differential equation and solve it.

In Section 3, we give some identities on higher order Bernoulli polynomials using ordinary differential equations.

2. Construction of nonlinear differential equations

We define that

$$B=B(t)=\frac{t}{1-e^t}. \tag{2.1}$$

By differentiating (2.1) with respect to t, we get

$$B^2=tB^{(1)}+(t-1)B. \tag{2.2}$$

By differentiating (2.2) with respect to t, we get

$$2B^3=t^2B^{(2)}+(3t^2-2t)B^{(1)}+(2t^2-3t+2)B. \tag{2.3}$$

Continuing this process, we get

$$N!B^{N+1}=\sum_{k=1}^{N+1}a_k(N,t)B^{(N-k+1)}=a_1(N,t)B^{(N)}+a_2(N,t)B^{(N-1)}+\cdots+a_N(N,t)B^{(1)}+a_{N+1}(N,t)B, \tag{2.4}$$

where $B^{(N)}=\frac{d^NB}{dt^N}$.

Let us consider the derivative of (2.4) with respect to t to find the recurrence relation of the coefficient $a_k(N,t)$ in (2.4). By differentiating (2.4) with respect to t and multiplying by t, we obtain

$$t(N+1)!B^NB^{(1)}=t\sum_{k=1}^{N+1}\left(a_k(N,t)B^{(N-k+2)}+\frac{d}{dt}a_k(N,t)B^{(N-k+1)}\right). \tag{2.5}$$

From (2.2), we have that the left hand side of (2.5) is

$$LHS=(N+1)!B^NtB^{(1)}=(N+1)!B^N\left(B^2+(1-t)B\right)=(N+1)!B^{N+2}-(N+1)(t-1)N!B^{N+1}, \tag{2.6}$$

and the right hand side of (2.5) is

$$RHS=\sum_{k=0}^{N}ta_{k+1}(N,t)B^{(N-k+1)}+\sum_{k=1}^{N+1}t\frac{d}{dt}a_k(N,t)B^{(N-k+1)}. \tag{2.7}$$

By (2.4), (2.6) and (2.7), we get