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The difference between the metric dimension and the determining number of a graph

Delia Garijo, Antonio González *, Alberto Márquez

Departamento de Matemática Aplicada I, Universidad de Sevilla, Spain

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ABSTRACT

We study the maximum value of the difference between the metric dimension and the determining number of a graph as a function of its order. We develop a technique that uses functions related to locating-dominating sets to obtain lower and upper bounds on that maximum, and exact computations when restricting to some specific families of graphs. Our approach requires very diverse tools and connections with well-known objects in graph theory; among them: a classical result in graph domination by Ore, a Ramsey-type result by Erdős and Szekeres, a polynomial time algorithm to compute distinguishing sets and determining sets of twin-free graphs, k-dominating sets, and matchings.

1. Introduction and preliminaries

Roughly speaking a *resolving set* is a subset of the vertices of a graph such that all other vertices are uniquely determined by their distances to those vertices. This concept was introduced in the 1970s by Harary and Melter [21], and independently by Slater [31]. Since one obtains a labeling process for all the vertices, resolving sets can be used to store the position of a mobile object in a scenario modeled by a graph, and design effective algorithms to robot navigation. This is not the only area where this type of sets can be used; we refer the reader to [6] and the survey of Bailey and Cameron [2] for more references on applications to coin weighing problems, strategies for Mastermind game, and pattern recognition, among others.

Obviously, in order to design effective algorithms, resolving sets are required to have a cardinality as small as possible but it is also important to consider the following property related to symmetries: the only automorphism of the graph fixing a resolving set is the identity. In general, it is possible to find subsets of vertices with this property (of "destroying" all the automorphisms) and with smaller cardinality than all the resolving sets in the graph; these are cases of *determining sets*, which were introduced in the 1970s by Sims [30] in the context of computational group theory as specific types of *bases*. Much later, Boutin [4] and Erwin and Harary [17] used respectively the terms determining set and *fixing set* to refer to the same concept.

In order to analyze how different resolving sets and determining sets can be, Boutin in [4] asked the following question on the parameters minimizing their cardinalities, which are formally defined below together with resolving sets and determining sets.

Problem 1. Can the difference between the determining number and the metric dimension of a graph be arbitrarily large?

* Corresponding author.
E-mail addresses: dgarijo@us.es (D. Garijo), gonzalezh@us.es (A. González), almar@us.es (A. Márquez).

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One of the main contributions of this paper is the technique that we have developed to approach this problem, which has interest by its own, since it combines very diverse tools that go from a classical result by Ore and a Ramsey-type result of Erdős and Szekeres to matchings and the design of a polynomial time algorithm to compute sets with some specific properties. To be more precise, we first provide some definitions and notations.

Let $G = (V(G), E(G))$ be a finite, simple, undirected, and connected graph of order $n = |V(G)|$. As usual, $\overline{G}$ denotes the complement of G. We write $N_G(u)$ and $N_G[u]$, respectively, for the open and the closed neighbourhood of a vertex $u \in V(G)$. The degree of vertex u is denoted by $\delta_G(u)$, and $\delta(G)$ is the minimum degree of G. The subscript G will be dropped from these notations when no confusion may arise.

An *automorphism* of G is a bijective mapping of $V(G)$ onto itself such that $f(u)f(v) \in E(G)$ if and only if $uv \in E(G)$. The *automorphism group* of G is denoted by $\mathrm{Aut}(G)$, and its identity element is id_G. The *stabilizer* of a set $S \subseteq V(G)$ is $\mathrm{Stab}(S) = \{\phi \in \mathrm{Aut}(G) | \phi(u) = u, \forall u \in S\}$, and S is a *determining set* of G if $\mathrm{Stab}(S) = \{id_G\}$. The minimum cardinality of a determining set is the *determining number* of G, written as $\mathrm{Det}(G)$.

The *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest u-v path. A vertex $u \in V(G)$ *resolves* a pair $\{x, y\} \subseteq V(G)$ if $d(u, x) \neq d(u, y)$. When every pair of vertices of G is resolved by some vertex in S, it is said that S is a *resolving set* of G. The minimum cardinality of a resolving set is the *metric dimension* of G, denoted by $\dim(G)$, and a resolving set of cardinality $\dim(G)$ is called a *metric basis* of G.

Problem 1 arises naturally since, as it was said before, every resolving set of a graph G is also a determining set, and so $\mathrm{Det}(G) \leqslant \dim(G)$ (see [4,17]). Further, the difference between both parameters is either zero or very small in many families of graphs; among them: paths, cycles, complete graphs, and 2-dimensional grids [17,27]. To approach the question we first define the function $(\dim - \mathrm{Det})(n)$ as the maximum value of $\dim(G) - \mathrm{Det}(G)$ over all graphs G of order n (note that its computation would give the answer to the problem). Then, we develop a technique based mainly on the study of two functions (which are introduced below) related to locating-dominating sets: $(\lambda - \mathrm{Det})(n)$ and $\lambda_{|_{\mathcal{C}^*}}(n)$. Besides its independent interest, this technique lets us improve significantly the best result known to date on Problem 1 which, in terms of our function $(\dim - \mathrm{Det})(n)$, is the following.

Proposition 1.1 [5]. *For every $n \geqslant 8$,*

$$\left\lfloor \frac{2}{5} n \right\rfloor - 2 \leqslant (\dim - \mathrm{Det})(n) \leqslant n - 2.$$

A vertex $u \in V(G)$ *distinguishes* a pair $\{x, y\} \subseteq V(G)$ if either $u \in \{x, y\}$ or precisely one of x, y is adjacent to u, and a set $D \subseteq V(G)$ is a *distinguishing set* of G if every pair of vertices of G is distinguished by some vertex in D. When D is also a *dominating set* (i.e., $N(x) \cap D \neq \emptyset$ for every $x \in V(G) \setminus D$) it is said that D is a *locating-dominating set*. The minimum cardinality of a locating-dominating set is the *locating-domination number* of G, denoted by $\lambda(G)$. Note that $\lambda(G) \leqslant n - 1$ since every subset of $n - 1$ vertices is a locating-dominating set of G.

Although distinguishing sets and locating-dominating sets were introduced in different contexts (see [1,32]) they are in essence the same concept: given a distinguishing set $D \subseteq V(G)$, by definition there is at most one vertex $x \in V(G) \setminus D$ so that $N(x) \cap D = \emptyset$. Thus $D \cup \{x\}$ is a locating-dominating set. This yields the following.

Observation 1.2. Let D be a distinguishing set of a graph G. Then, $\lambda(G) \leqslant |D| + 1$.

Every locating-dominating set $D \subseteq V(G)$ is clearly a resolving set since each pair $\{x, y\} \subseteq V(G) \setminus D$ is distinguished by some vertex $u \in D$ and so either $d(u, x) = 1 < d(u, y)$ or $d(u, y) = 1 < d(u, x)$. Thus, $\mathrm{Det}(G) \leqslant \dim(G) \leqslant \lambda(G)$ for every graph G.

Let $(\lambda - \mathrm{Det})(n)$ and $\lambda(n)$ be the maximum values of, respectively, $\lambda(G) - \mathrm{Det}(G)$ and $\lambda(G)$ over all graphs G of order n. Note that the function $\lambda(n)$ equals $n - 1$ (attained by the complete graph K_n) but the non-trivial restriction of this function to the class $\mathcal{C}^*$ of *twin-free* graphs (i.e., graphs that do not contain *twin* vertices, which are formally defined in SubSection 3.1), denoted by $\lambda_{|_{\mathcal{C}^*}}(n)$, will play an important role throughout the paper. Thus,

$$(\dim - \mathrm{Det})(n) \leqslant (\lambda - \mathrm{Det})(n) \leqslant \lambda(n) = n - 1. \tag{1}$$

In Section 2, we find lower bounds on the functions $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$ by constructing appropriate families of graphs. In particular, we improve the lower bound of Proposition 1.1 and conjecture that these new bounds are precisely the exact expressions of those functions.

Section 3 develops a method to prove that $\lambda_{|_{\mathcal{C}^*}}(n)$ is an upper bound on $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$, which is a key result in our study. Moreover, we conjecture a formula for the function $\lambda_{|_{\mathcal{C}^*}}(n)$.

Sections 4 and 5 contain two explicit upper bounds on $\lambda_{|_{\mathcal{C}^*}}(n)$. Although the one in Section 5 gives a better approach, we believe that the technique used to obtain the bound in Section 4 has interest by its own and so it is worth to be included in this paper. This technique uses a variant of a classical theorem in domination theory due to Ore [28], which lets us relate, for twin-free graphs, the locating-domination number with a series of classical graph parameters (following the same spirit as the relationships existing among different domination parameters; see [23] for a number of examples). The desired bound is then obtained by using those relations and a Ramsey-type result of Erdős and Szekeres [16].