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A Philos-type theorem for third-order nonlinear retarded dynamic equations

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ABSTRACT

A new Philos-type theorem for a class of third-order nonlinear dynamic equations is presented. Among others, the restrictive condition in terms of the commutativity of the jump and delay operators is removed here.

1. Introduction

This paper is concerned with oscillation and asymptotic behavior of solutions to a third-order nonlinear delay dynamic equation

$$\left(a((rx^{\Delta})^{\Delta})^{\gamma}\right)^{\Delta}(t)+f(t,x(\tau(t)))=0, \tag{1.1}$$

where $\gamma>0$ is the ratio of odd positive integers, $1/a$ and $1/r$ are positive rd-continuous functions defined on $\mathbb{T}$, $\tau\in C_{\mathrm{rd}}(\mathbb{T},\mathbb{T}), \tau(t)\leqslant t, \lim_{t\to\infty}\tau(t)=\infty, f\in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$ is assumed to satisfy $uf(t,u)>0$ for $u\neq 0$, and there exists a positive rd-continuous function p defined on $\mathbb{T}$ such that $f(t,u)/u^{\gamma}\geqslant p(t)$ for $u\neq 0$.

Since we are interested in oscillatory and asymptotic properties, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above. Also we assume $t_0\in\mathbb{T}$ and it is convenient to assume $t_0>0$, and define the time scale interval of the form $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$. By a solution of (1.1) we mean a real-valued function $x\in C^1_{\mathrm{rd}}[T_x,\infty)_{\mathbb{T}}$, $T_x\in[t_0,\infty)_{\mathbb{T}}$ which has the properties that $rx^{\Delta}\in C^1_{\mathrm{rd}}[T_x,\infty)_{\mathbb{T}}$, $a((rx^{\Delta})^{\Delta})^{\gamma}\in C^1_{\mathrm{rd}}[T_x,\infty)_{\mathbb{T}}$, and satisfies (1.1) on $[T_x,\infty)_{\mathbb{T}}$. We consider only proper solutions x to (1.1) that satisfy $\sup\{|x(t)|: t\in[T,\infty)_{\mathbb{T}}\}>0$ for all sufficiently large $T\in[T_x,\infty)_{\mathbb{T}}$, and we tacitly assume that such solutions exist. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is termed nonoscillatory.

Following Hilger's landmark contribution [15], the theory of time scales has recently received a lot of attention. Several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal et al. [3] and the references cited therein. A book on the subject of time scales, by Bohner and Peterson [7], summarizes and organizes much of the time scale calculus; we also refer to the book by Bohner and Peterson [8] for advances in dynamic equations on time scales.

In recent years, there has been much research activity concerning oscillation and asymptotic behavior of different classes of dynamic equations on time scales, we refer the reader to [1,4,5,9,11–14,17–20,23–25] and the references cited therein.

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The analogue for (1.1) and its particular case where $\mathbb{T}=\mathbb{R}$ has been studied in [2,6,10,21]. Thereinto, Hassan [14] established several criteria for (1.1) in the case where

$$\gamma \geqslant 1, \quad \tau\circ\sigma=\sigma\circ\tau, \quad \text{and} \quad \tau^{\Delta}>0. \tag{1.2}$$

Agarwal et al. [4] obtained some new results for (1.1) without requiring (1.2), one of which we present below for the convenience of the reader. In what follows, we use the notation

$$\alpha_{+}^{\Delta}(t):=\max\{0,\alpha^{\Delta}(t)\}, \quad \varphi(t):=\begin{cases}\phi(t), & \text{if } 0<\gamma\leq 1,\\ \phi^{\gamma}(t), & \text{if } \gamma>1,\end{cases}$$

and

$$\phi(t):=\frac{m(t)}{m(\sigma(t))}, \quad \delta(t,u):=\int_u^t \frac{\Delta s}{a^{\frac{1}{\gamma}}(s)},$$

where m is an auxiliary function that will be specified later.

Theorem 1.1 (See [4, Theorem 3.5]). *Assume*

$$\int_{t_0}^{\infty}\frac{\Delta t}{a^{\frac{1}{\gamma}}(t)}=\infty, \quad \int_{t_0}^{\infty}\frac{\Delta t}{r(t)}=\infty, \tag{1.3}$$

and

$$\int_{t_0}^{\infty}\frac{1}{r(t)}\int_t^{\infty}\left[\frac{1}{a(s)}\int_s^{\infty}p(u)\Delta u\right]^{\frac{1}{\gamma}}\Delta s\Delta t=\infty. \tag{1.4}$$

Suppose also that there exist a positive function $m\in C^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ and a $T_1\in[t_*,\infty)_{\mathbb{T}}$ such that

$$\frac{m(t)}{\delta(t,t_*)a^{\frac{1}{\gamma}}(t)}-m^{\Delta}(t)\leq 0 \quad \text{for} \quad t\in[T_1,\infty)_{\mathbb{T}}. \tag{1.5}$$

If there exists a positive function $\alpha\in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that for all sufficiently large $T_1\in[t_0,\infty)_{\mathbb{T}}$ and for some $T\in[T_1,\infty)_{\mathbb{T}}$,

$$\limsup_{t\to\infty}\int_T^t\left[\alpha^{\sigma}(s)p(s)M(s,T_1)-\frac{a(s)(\alpha_{+}^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\varphi(s)\alpha^{\sigma}(s))^{\gamma}}\right]\Delta s=\infty, \tag{1.6}$$

where

$$M(t,T_1):=\left(\frac{1}{m(\sigma(t))}\int_{T_1}^{\tau(t)}\frac{m(s)}{r(s)}\Delta s\right)^{\gamma},$$

then every solution of (1.1) is either oscillatory or tends to zero as $t\to\infty$.

As is well known, Kamenev-type and Philos-type criteria are important breakthroughs in the development of oscillation theory of differential equations; see [16,22]. Agarwal et al. [5] obtained Philos-type oscillation criteria for a second-order half-linear dynamic equation on time scales. Note that the paper [4] cannot provide Philos-type criteria for (1.1). Hence, the objective of this paper is to give a Philos-type criterion for Eq. (1.1).

This paper is organized as follows: In Section 2, we employ the Riccati transformation technique to derive our main results. In Section 3, we provide some discussions to summarize the contents of this paper. In what follows, all functional inequalities are assumed to hold eventually, that is, they are satisfied for all t large enough.

2. Main results

We now present the main results. For the sake of convenience, we use the notation

$$\mathbb{D}\equiv\{(t,s): t_0\leqslant s\leqslant t, \quad t,s\in[t_0,\infty)_{\mathbb{T}}\}$$

and

$$\mathbb{D}_0\equiv\{(t,s): t_0\leqslant s< t, \quad t,s\in[t_0,\infty)_{\mathbb{T}}\}.$$

We say that a function $H\in C_{\mathrm{rd}}(\mathbb{D},[0,\infty))$ belongs to a class $\mathcal{H}$ if

(*i*) $H(t,t)=0$ for $t\in[t_0,\infty)_{\mathbb{T}}$ and $H(t,s)>0$ for $(t,s)\in\mathbb{D}_0$;
(*ii*) H has a nonpositive rd-continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ on $\mathbb{D}_0$ with respect to the second variable.