Applied Mathematics and Computation 249 (2014) 535–540

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Δ-convergence theorems for multi-valued nonexpansive mappings in hyperbolic spaces ☆

Shih-sen Chang [a,*], Gang Wang [a], Lin Wang [a], Yong Kun Tang [a], Zhao Li Ma [b]

[a] College of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan 650221, China
[b] School of Information Engineering, The College of Arts and Sciences, Yunnan Normal University, Kunming, Yunnan 650222, China

ARTICLE INFO

Keywords:
Hyperbolic space
Multi-valued nonexpansive mapping
Common fixed point
Δ-convergence theorem

ABSTRACT

The purpose of this paper is to introduce the *mixed Agarwal–O'Regan–Sahu type iterative scheme* (Agarwal et al., 2007) for finding a common fixed point of the multi-valued nonexpansive mappings in the setting of *hyperbolic spaces*. Under suitable conditions, some Δ-convergence theorems of the iterative sequence generated by the proposed scheme to approximate a common fixed point of multi-valued nonexpansive mappings are proved. The results presented in the paper extend and improve some recent results announced in the current literature.

1. Introduction and preliminaries

Throughout this paper, we work in the setting of *hyperbolic spaces* introduced by Kohlenbach [1].

A *hyperbolic space* is a metric space (X,d) with a mapping $W : X^2 \times [0,1] \to X$ satisfying the following conditions.

(i) $d(u, W(x,y,\alpha)) \leqslant (1-\alpha)d(u,x) + \alpha d(u,y)$;
(ii) $d(W(x,y,\alpha), W(x,y,\beta)) = |\alpha-\beta| d(x,y)$;
(iii) $W(x,y,\alpha) = W(y,x,(1-\alpha))$;
(iv) $d(W(x,z,\alpha), W(y,w,\alpha)) \leqslant (1-\alpha)d(x,y) + \alpha d(z,w)$.

for all $x,y,z,w \in X$ and $\alpha,\beta \in [0,1]$.

A nonempty subset K of a hyperbolic space X is said to be *convex*, if $W(x,y,\alpha) \in K$ for all $x,y \in K$ and $\alpha \in [0,1]$.

Remark. It should be pointed out that if a metric space (X,d) with a mapping $W : X^2 \times [0,1] \to X$ satisfying only condition (i), then it coincides with the *convex metric space* introduced by Takahashi [2]. The concept of hyperbolic spaces given here is more restrictive than the *hyperbolic type* defined by Goebel and Kirk [3], since the conditions (i)–(iii) together are equivalent to (X, d) being a space of hyperbolic type in [3]. But it is slightly more general than the hyperbolic space defined in Reich and Shafrir [4]. The class of hyperbolic spaces contains normed linear spaces and convex subsets, therefore the Hilbert ball equipped with the hyperbolic metric [5], $\mathbb{R}$-*trees*, *Hadamard manifolds* as well as *CAT(0) spaces* in the sense of Gromov (see [6]).

☆ Supported by the National Natural Science Foundation of China (Grant No. 11361070).
* Corresponding author.
E-mail address: changss2013@aliyun.com (S.-s. Chang).

http://dx.doi.org/10.1016/j.amc.2014.10.076
0096-3003/

A hyperbolic space is said to be *uniformly convex* [7] if for any $r > 0$ and $\epsilon \in (0,2]$, there exists a $\delta \in (0,1]$ such that for all $u,x,y \in X$,

$$d\left(W\left(x,y,\frac{1}{2}\right),u\right) \leqslant (1-\delta)r,$$

provided $d(x,u) \leqslant r,\ d(y,u) \leqslant r$ and $d(x,y) \geqslant \epsilon r$.

A map $\eta : (0,\infty) \times (0,2] \to (0,1]$ which provides such a $\delta = \eta(r,\epsilon)$ for given $r > 0$ and $\epsilon \in (0,2]$, is known as a modulus of uniform convexity of X. η is said to be monotone, if it decreases with r (for a fixed ϵ), *i.e.*, $\forall \epsilon > 0, \forall r_2 \geqslant r_1 > 0\,(\eta(r_2,\epsilon) \leqslant \eta(r_1,\epsilon))$.

In the sequel, let (X,d) be a metric space and let K be a nonempty subset of X . We shall denote the fixed point set of a mapping T by $F(T) = \{x \in K : Tx = x\}$.

K is called *proximal*, if for each $x \in X$, there exists an element $y \in K$ such that

$$d(x,y) = d(x,K) := inf_{z\in K} d(x,z).$$

In the sequel, we denote by $C(K)$ the collection of all nonempty compact subsets of K, and denote by $CB(K)$, and $P(K)$ the collection of all nonempty closed bounded subsets, and nonempty proximal bounded subsets of K, respectively. The *Hausdorff metric H* on $CB(X)$ is defined by

$$H(A,B) := \max\left\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\right\}, \quad \forall A,\ B \in CB(X)$$

Let $T : K \to CB(X)$ be a multi-valued mapping. An element $x \in K$ is said to be a fixed point of T, if $x \in Tx$. A multi-valued mapping $T : K \to CB(X)$ is said to be *nonexpansive*, if

$$H(Tx,Ty) \leqslant d(x,y), \quad \forall x,y \in K.$$

The existence of fixed points of various nonlinear mappings has relevant applications in many branches of nonlinear analysis and topology. On the other hand, there are certain situations where it is difficult to derive conditions for the existence of fixed points for certain types of nonlinear mappings. It is worth to mentioning that fixed point theory for nonexpansive mappings requires tools far beyond from metric fixed point theory. Iteration schemas are the only main tool for analysis of generalized nonexpansive mappings. Fixed point theory has a computational flavor as one can define effective iteration schemas for the computation of fixed points of various nonlinear mappings. The problem of finding a common fixed point of some nonlinear mappings acting on a nonempty convex domain often arises in applied mathematics.

The purpose of this paper is to introduce the *mixed Agarwal–O'Regan–Sahu type iterative scheme* [8] for finding a common fixed point of the multi-valued nonexpansive mappings in the setting of *hyperbolic spaces*. Under suitable conditions some Δ-convergence theorems of the iterative sequence generated by the proposed scheme to approximate a common fixed point of multi-valued nonexpansive mappings are proved. The results presented in the paper extend and improve some recent results announced in the current literature [7–22].

In order to define the concept of Δ-*convergence* in the general setting of hyperbolic spaces [23], we first recall some definitions, notations, and conclusions which will be needed in proving our main results.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space (X,d). For $x \in X$, we define a continuous functional $r(\cdot,\{x_n\}) : X \to [0,\infty)$ by

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n). \tag{1.1}$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\}. \tag{1.2}$$

The *asymptotic center* $A_K(\{x_n\})$ of a bounded sequence $\{x_n\}$ with respect to $K \subset X$ is the set

$$A_K(\{x_n\}) = \{x \in X : r(x,\{x_n\}) \leqslant r(y,\{x_n\}), \quad \forall y \in K\}. \tag{1.3}$$

This implies that the asymptotic center is the set of minimizer of the functional $r(\cdot,\{x_n\})$ in K. If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$. It is known that uniformly convex Banach spaces and CAT(0) spaces enjoy the property that "bounded sequences have unique asymptotic centers with respect to closed convex subsets". The following Lemma is due to Leustean [25] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 1.1 [25]. *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in K has a unique asymptotic center in K.*

Recall that a sequence $\{x_n\}$ in X is said to Δ-*converge to* $x \in X$, if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the Δ – *limit of* $\{x_n\}$.