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Construction of energy-stable projection-based reduced order models

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A B S T R A C T

An approach for building energy-stable Galerkin reduced order models (ROMs) for linear hyperbolic or incompletely parabolic systems of partial differential equations (PDEs) using continuous projection is developed. This method is an extension of earlier work by the authors specific to the equations of linearized compressible inviscid flow. The key idea is to apply to the PDEs a transformation induced by the Lyapunov function for the system, and to build the ROM in the transformed variables. For linear problems, the desired transformation is induced by a special inner product, termed the "symmetry inner product", which is derived herein for several systems of physical interest. Connections are established between the proposed approach and other stability-preserving model reduction methods, giving the paper a review flavor. More specifically, it is shown that a discrete counterpart of this inner product is a weighted L^2 inner product obtained by solving a Lyapunov equation, first proposed by Rowley et al. and termed herein the "Lyapunov inner product". Comparisons between the symmetry inner product and the Lyapunov inner product are made, and the performance of ROMs constructed using these inner products is evaluated on several benchmark test cases.

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1. Introduction

Numerous modern-day engineering problems require the simulation of complex systems with tens of millions or more unknowns. Despite improved algorithms and the availability of massively parallel computing resources, "high-fidelity" models are, in practice, often too computationally expensive for use in a design or analysis setting. The continuing push to incorporate into modeling efforts the quantification of uncertainties, critical to many science and engineering applications, can present an intractable computational burden due to the high-dimensional systems that arise. This situation has prompted researchers to develop reduced order models (ROMs): models constructed from high-fidelity simulations that retain the essential physics and dynamics of their corresponding full order models (FOMs), but have a much lower computational cost. Since ROMs are, by construction, small, they can enable uncertainty quantification (UQ) as well as on-the-spot decision making and/or control.

In order to serve as a useful predictive tool, a ROM should possess the following properties: consistency (with respect to its corresponding high-fidelity model), stability, and convergence (to the solution of its corresponding high-fidelity model).

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The second of these properties, namely numerical stability, is particularly important, as it is a prerequisite for studying the convergence and accuracy of a ROM. It is well-known that popular model reduction approaches known as the proper orthogonal decomposition (POD) method [24,25,18] and the balanced proper orthogonal decomposition (BPOD) method [30,22] lack, in general, an *a priori* stability guarantee. In [29], Amsallem et al. suggest that POD ROMs constructed for linear time-invariant (LTI) systems in descriptor form tend to possess better numerical stability properties than POD ROMs constructed for LTI systems in non-descriptor form. Although heuristics such as these exist, it is in general unknown *a priori* if a ROM constructed using POD or BPOD will preserve the stability properties of the high-fidelity system from which the model was constructed. There *does* exist a model reduction technique that has a rigorous stability guarantee, namely balanced truncation [28,10]; however, the computational cost of this method, which requires the computation and simultaneous diagonalization of infinite controllability and observability Gramians, makes balanced truncation computationally intractable for systems of very large dimensions (i.e., systems with more than 10,000 degrees of freedom [23]).

The importance of obtaining stable ROMs has been recognized in recent years by a number of authors. It is shown by Patera, Veroy and Rozza in [26,27] that a stable ROM can be constructed using the reduced basis method. In [23], Rowley et al. show that Galerkin projection preserves the stability of an equilibrium point at the origin if the ROM is constructed in an "energy-based" inner product. In [6,7], Barone et al. demonstrate that a symmetry transformation leads to a stable formulation for a Galerkin ROM for the linearized compressible Euler equations [6,7] with solid wall and far-field boundary conditions. In [1], Serre et al. propose applying the stabilizing projection developed by Barone et al. in [6,7] to a skew-symmetric system constructed by augmenting a given linear system with its adjoint system. This approach yields a ROM that is stable at finite time even if the solution energy of the full-order model is growing.

The methods described above derive (*a priori*) a stability-preserving model reduction framework that is specific to a particular equation set. There exist, in addition to these techniques, approaches which stabilize an unstable ROM through a post-processing (*a posteriori*) stabilization step applied to an unstable algebraic ROM system. Ideally, the stabilization is such that it will only minimally modify the ROM. In [5], Amsallem et al. propose a method for stabilizing projection-based linear ROMs through the solution of a small-scale convex optimization problem. In [37], a set of linear constraints for the left-projection matrix, given the right-projection matrix, are derived by Bond et al. to yield a projection framework that is guaranteed to generate a stable ROM. An approach for stabilizing unstable ROMs for LTI systems, termed ROM stabilization via optimization-based eigenvalue reassignment, was proposed by Kalashnikova et al. in the recent work [53]. In this approach, the unstable eigenvalues of an unstable ROM are modified through the numerical solution of a constrained nonlinear least-squares optimization problem formulated such that the error in the stabilized ROM output is minimal. In [38], a ROM stabilization methodology that achieves improved accuracy and stability through the use of a new set of basis functions representing the small, energy-dissipation scales of turbulent flows is derived by Balajewicz et al. In [34], Zhu et al. derive some large eddy simulation (LES) closure models for POD ROMs for the incompressible Navier–Stokes equations, and demonstrate numerically that the inclusion of these LES terms yields a ROM with increased numerical stability (albeit at the sacrifice of consistency of the ROM with respect to the direct numerical simulation (DNS) from which the ROM is constructed).

In this article, several approaches to building stable ROMs for linear systems, both in the continuous as well as in the discrete projection setting, are presented, connected and extended. The article has a review flavor, but contains several new contributions, most notably the following:

- The energy-stable continuous projection ROM method developed specifically for the equations of linearized compressible inviscid flow in [6,7] is extended to generic systems of PDEs of the hyperbolic or incompletely parabolic type.
- A stability preserving symmetry inner product is derived for several physical systems (the wave equation, the linearized shallow water equations, the linearized compressible Euler equations, the linearized compressible Navier–Stokes equations).
- Connections between the proposed energy-stable continuous projection method and other model reduction techniques with an *a priori* stability guarantee, e.g., a discrete projection approach involving a Lyapunov equation-based inner product introduced by Rowley et al. in [23], are established using the concept of energy stability.
- Numerical studies evaluating the performance of ROMs constructed in the energy inner products described herein are performed.

The remainder of this paper is organized as follows. The first part consists of some preliminaries: projection-based model reduction (in particular, the POD[1]/Galerkin method) is overviewed (Section 2), and several notions of stability (energy-stability, Lyapunov stability, asymptotic stability, exponential stability, time-stability) are defined (Section 3). Attention is then turned to the construction of energy-stable ROMs for linear systems of PDEs using continuous projection (Section 4). The energy-stability preserving model reduction approach developed specifically for the equations of linearized compressible inviscid flow in [6,7] is generalized. Examples of this inner product are given for several systems of physical interest, and some numerical results are presented. Next, it is shown that a certain transformation applied to a generic linear hyperbolic or incompletely parabolic set of PDEs and induced by the Lyapunov function for these equations will yield a Galerkin ROM that is stable for *any* choice of

[1] For concreteness, it is assumed herein that the reduced basis is constructed via the POD method, as the POD is a popular method for computing reduced bases that is feasible even for very large systems but can give rise to unstable ROMs. It is emphasized that the energy-stability results discussed herein hold for *any* choice of reduced basis, not just the POD basis, however.