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A note on a fast breakdown-free algorithm for computing the determinants and the permanents of k-tridiagonal matrices

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ABSTRACT

k-Tridiagonal matrices have attracted much attention in recent years, which are a generalization of tridiagonal matrices. In this note, a breakdown-free numerical algorithm of $\mathcal{O}(n)$ is presented for computing the determinants and the permanents of k-tridiagonal matrices. Even though the algorithm is not a symbolic algorithm, it never suffers from breakdown. Furthermore, it produces exact values when all entries of the k-tridiagonal matrices are given in integer. In addition, the algorithm can be simplified for a general symmetric Toeplitz case, and it generates the kth powers of Fibonacci, Pell, and Jacobsthal numbers for a certain symmetric Toeplitz case.

1. Introduction

We consider the determinants and the permanents of the n-square k-tridiagonal matrix [16] of the form:

$$T_n^{(k)} = \begin{bmatrix} d_1 & 0 & \cdots & 0 & a_1 & 0 & \cdots & 0 \\ 0 & d_2 & & & & a_2 & & \vdots \\ \vdots & & \ddots & & & & \ddots & 0 \\ 0 & & & d_{n-k} & & & & a_{n-k} \\ b_{k+1} & & & & \ddots & & & 0 \\ 0 & b_{k+2} & & & & \ddots & & \vdots \\ \vdots & & \ddots & & & & d_{n-1} & 0 \\ 0 & \cdots & 0 & b_n & 0 & \cdots & 0 & d_n \end{bmatrix}. \quad (1)$$

The k-tridiagonal matrix includes some important classes of matrices such as a tridiagonal matrix and a constant-diagonals matrix [12,17], and some properties of the matrix have been found in [9,11,16].

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The determinant and the permanent of $n \times n$ matrix $A = (a_{ij})$ are defined as

$$\det(A) := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}, \quad \mathrm{per}(A) := \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n represents the symmetric group of degree n, and it is well known that the exact computation of the permanent is in general difficult, see e.g. [5] and references therein.

For computing the determinant of the matrix (1), a breakdown-free algorithm has very recently been found by El-Mikkawy [7], and then the algorithm has been used for computing the inversion [8]. Since the algorithm is a symbolic algorithm, it does not suffer from breakdown. On the other hand, it may be time-consuming work when many symbolic names arise during the symbolic computation. This motivates us to find a breakdown-free algorithm without using symbolic computation. For related work, see [18] for the determinant based on the LU factorization and [1] for the determinant and the permanent of the special case, i.e., k-tridiagonal Toeplitz matrix. Recent developments of algorithms for the determinants of other matrices, see e.g. [2–4,6,10,13–15].

In this note, without using symbolic computation, a breakdown-free algorithm is presented for the determinant and the permanent of the matrix (1).

2. Main result

Let $[j]$ be the equivalence class of the form $[j] := \{i \in \mathbb{N}_n | i \equiv j \pmod{k}\}$, where $\mathbb{N}_n := \{i\}_{1 \leqslant i \leqslant n}$, and let $|[j]|$ be the number of elements of $[j]$. Then, Algorithm 1 can be used for computing the determinant and the permanent of the k-tridiagonal matrix.

Algorithm 1. An $\mathcal{O}(n)$ algorithm for $\det(T_n^{(k)})$ and $\mathrm{per}(T_n^{(k)})$

set $p = -1$ for the determinant or $p = 1$ for the permanent.
for $j = 1, 2, \ldots, k$ **do:**
  $c_j = d_j$
  $c_{k+j} = d_{k+j} \cdot c_j + p \cdot b_{k+j} \cdot a_j$
  for $i = 3, 4, \ldots, |[j]|$ **do:**
  $c_{k(i-1)+j} = d_{k(i-1)+j} \cdot c_{k(i-2)+j} + p \cdot b_{k(i-1)+j} \cdot a_{k(i-2)+j} \cdot c_{k(i-3)+j}$
end for
end for

After running Algorithm 1, the determinant and the permanent are given by

$$\begin{cases} \det(T_n^{(k)}) = \prod_{i=1}^{k} c_{n-k+i} & (\text{if } p = -1), \\ \mathrm{per}(T_n^{(k)}) = \prod_{i=1}^{k} c_{n-k+i} & (\text{if } p = 1). \end{cases} \tag{2}$$

It may be interesting to see that Algorithm 1 has no division. Thus the algorithm never fails, even though it is not a symbolic algorithm. Furthermore, the algorithm exactly produces the determinant and the permanent if all entries of $T_n^{(k)}$ are given in integer, which will be seen in the next section. The proof of Algorithm 1 is given in Appendix A.

We now consider the special case where $d_i = a$ and $a_i = b_i = b$ for all i, and we assume that there exists a number m such that $n = mk$, i.e., $T_n^{(k)}$ to be k-tridiagonal symmetric Toeplitz matrix with the relation $n = mk$. The resulting matrix is denoted by $H_{n,k}(a, b)$. In this case, since $|[j]| = m$ for all j, Algorithm 1 is simplified. The resulting algorithm is described in Algorithm 2.

Algorithm 2. An $\mathcal{O}(n)$ algorithm for $\det(H_{n,k}(a, b))$ and $\mathrm{per}(H_{n,k}(a, b))$

set $p = -1$ for the determinant or $p = 1$ for the permanent.
set $u_1 = a$
set $u_2 = a \cdot u_1 + p \cdot b^2$
for $i = 3, 4, \ldots, m$ **do:**
  $u_i = a \cdot u_{i-1} + p \cdot b^2 \cdot u_{i-2}$
end for