Applied Mathematics and Computation 247 (2014) 1–12

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Cross-diffusion induced instability and pattern formation for a Holling type-II predator–prey model ☆

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ARTICLE INFO

Keywords:
Predator–prey model
Cross-diffusion
Instability
Pattern formation
Holling type-II

ABSTRACT

This paper is devoted to studying a predator–prey model with Holling type-II functional response and cross-diffusion subject to Neumann boundary condition. Our main interest lies in the effects of cross-diffusion on stability and stationary patterns. More precisely, the presented results show that cross-diffusion can *not only* destabilize a uniform equilibrium which is stable for the kinetic and random diffusion reaction systems, *but also* create spatial patterns even when the random diffusion fails to do so. Furthermore, our results also reveal that, in this kind of ecological system, instability and stationary patterns can appear only when the predators rapidly move away from a large group of preys, regardless of the speed that the preys keep away from the predators.

1. Introduction

In population dynamics, the spatially homogeneous predator–prey system with Holling type-II functional response has been extensively studied in the existing literature, see for example [1–5]. This model takes the following form:

$$\begin{cases} \frac{dU}{dt} = \beta U\left(1-\frac{U}{K}\right) - \frac{EMUV}{A+U}, \\ \frac{dV}{dt} = V\left(\frac{MU}{A+U} - D\right), \end{cases} \tag{1.1}$$

where U and V represent the densities of the prey and the predator, respectively; β, A, D, E, K, M are positive constants. β, K and E represent the intrinsic growth rate, the carrying capacity and the relative loss of the prey, respectively; D is the death rate of the predator; the function $\frac{MU}{A+U}$ is the Holling type-II functional response of the predator.

For simplicity, applying the following scaling to (1.1)

$$t = \beta\bar{t}, \quad u = U, \quad v = \frac{EM}{\beta}V,$$

then system (1.1) becomes

☆ Supported by the NNSF of China (Nos. 11271379, 11271381) and the National Basic Research Program of China (973 Program) (Grant No. 2010CB808002).

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http://dx.doi.org/10.1016/j.amc.2014.08.088
0096-3003/

$$\begin{cases} \frac{du}{dt} = u\left(1 - \frac{u}{k}\right) - \frac{uv}{a+u}, \\ \frac{dv}{dt} = v\left(\frac{bu}{a+u} - c\right), \end{cases} \tag{1.2}$$

where $a = A, k = K, b = \frac{M}{\beta}, c = \frac{D}{\beta}$. Moreover, if $b > \frac{c(k+a)}{k}$, then the unique positive equilibrium $\mathbf{u}^* = (u^*, v^*)^T$ of (1.2) is given by

$$u^* = \frac{ac}{b-c}, \quad v^* = \frac{(a+u^*)(k-u^*)}{k}.$$

When the densities of the prey and predator are spatially inhomogeneous in a bounded domain with smooth boundary $\Omega \subset \mathbf{R}^N (N \geqslant 1)$, instead of the ordinary differential system (1.2), one gets the following reaction–diffusion system

$$\begin{cases} u_t - d_1 \Delta u = u\left(1 - \frac{u}{k}\right) - \frac{uv}{a+u}, & \text{in } \Omega \times (0, \infty), \\ v_t - d_2 \Delta v = v\left(\frac{bu}{a+u} - c\right), & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & \text{on } \overline{\Omega}. \end{cases} \tag{1.3}$$

Here, a, b, c, k are positive constants, ν is the outward unit normal vector on $\partial\Omega$. The positive constants d_1 and d_2 are called random diffusion coefficients, which represent the natural dispersive force of movement of the prey and predator, respectively. The homogeneous Neumann boundary condition means that the two species have zero flux across the boundary $\partial\Omega$. The admissible initial data $u_0(x)$ and $v_0(x)$ are nonnegative smooth functions which are not identically zero. For system (1.3), the properties of solutions such as stability, bifurcation and spatiotemporal patterns have been well researched (see e.g. [6–9]). We also mention that for (1.3) with Dirichlet boundary condition, many mathematical results have been obtained and we refer to [10–12] and references therein.

In the present paper, we shall introduce the random diffusion and cross-diffusion to (1.2) and study the following strongly coupled reaction–diffusion system

$$\begin{cases} u_t - \Delta(d_1 u + \rho_{12} uv) = u\left(1 - \frac{u}{k}\right) - \frac{uv}{a+u}, & \text{in } \Omega \times (0, \infty), \\ v_t - \Delta(d_2 v + \rho_{21} uv) = v\left(\frac{bu}{a+u} - c\right), & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & \text{on } \overline{\Omega} \end{cases} \tag{1.4}$$

and the associated steady state problem which satisfies

$$\begin{cases} -\Delta(d_1 u + \rho_{12} uv) = u\left(1 - \frac{u}{k}\right) - \frac{uv}{a+u}, & \text{in } \Omega, \\ -\Delta(d_2 v + \rho_{21} uv) = v\left(\frac{bu}{a+u} - c\right), & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where the diffusion term is $\Delta \boldsymbol{\phi}(\mathbf{u})$,

$$\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \boldsymbol{\phi}(\mathbf{u}) = \begin{bmatrix} d_1 u + \rho_{12} uv \\ d_2 v + \rho_{21} uv \end{bmatrix}. \tag{1.6}$$

The non-negative constants ρ_{12} and ρ_{21} are usually referred as cross-diffusion pressures, ρ_{12} represents the tendency that the prey keeps away from the predator. In a certain kind of predator–prey relationships, a large number of prey species form a huge group to protect themselves from the attack of predator. Thus ρ_{21} means the tendency of predators to move away from a large group of preys, see [13,14] for more ecological background with respect to cross diffusion.

In Turing's seminal paper [15], diffusion has been regarded as the driving force of the spontaneous emergence of spatio-temporal structure in a variety of non-equilibrium situations. There have been many works on the role of random diffusion (see e.g. [7,16–25]). Following Turing's idea, Shi et al. [26] further explored Turing's diffusion induced instability for the cross-diffusion systems. They showed that cross-diffusion can destabilize a uniform equilibrium which is stable for the kinetic and random diffusion reaction systems; on the other hand, cross-diffusion can also stabilize a uniform equilibrium which is stable for the kinetic system but unstable for the random diffusion reaction system.

Recently, Xie [27] studied system (1.4) but with diffusion term $\Delta \boldsymbol{\Psi}(\mathbf{u})$, where $\boldsymbol{\Psi}(\mathbf{u}) := (d_1 u(1 + d_3 v), d_2 v(1 + \frac{d_4}{1+u}))^T, d_3 > 0$ and $d_4 > 0$ represent cross-diffusion pressures which imply that the prey runs away from predator and the predator chases the prey (see e.g. [14,28]). The author showed that if the positive equilibrium solution $\mathbf{u}^*$ is linearly stable with respect to the ODE system (1.2), then it is also linearly stable for (1.4) with the diffusion term $\Delta \boldsymbol{\Psi}(\mathbf{u})$. That is to say, random diffusion and cross-diffusion can not drive instability (see [27, Theorem 1.1]). In [29], Feng and Wang investigated the associated steady state problem of (1.4) with the same diffusion term $\Delta \boldsymbol{\Psi}(\mathbf{u})$. They derived the existence of non-constant positive steady state when the positive equilibrium for the ODE system (1.2) is linearly unstable (see [[29]Theorem 6]). In [30], Liu and Lin considered a predator–prey model with Holling type III response function and cross-diffusion. In [31], Zhou and Kim investigated a Lotka–Volterra prey-predator model with cross-diffusion and Holling type-II functional response subject to