Applied Mathematics and Computation 247 (2014) 18–29

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Approximate solution of three-point boundary value problems for second-order ordinary differential equations with variable coefficients

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ARTICLE INFO

Keywords:
Three-point boundary value problems
Variable coefficients
Fredholm integral equation
Approximate solution
Convergence and error estimate

ABSTRACT

Some three-point boundary value problems for a second-order ordinary differential equation with variable coefficients are investigated in the present paper. By using the integration method, the second-order three-point boundary value problems are transformed into a Fredholm integral equation of the second kind. The solutions and Green's functions for some special cases of the second-order three-point boundary value problems can be determined easily. The existence and uniqueness of the solutions of the given Fredholm integral equations are considered by using the fixed point theorem in Banach spaces. A new numerical method is further proposed to solve the second-kind Fredholm integral equation and an approximate solution is made. The convergence and error estimate of the obtained approximate solution are further analyzed. Numerical results are carried out to verify the feasibility and novelty of the proposed solution procedures.

1. Introduction

Multipoint boundary value problems for ordinary differential equations can arise in solving linear partial differential equations by using the separation variable method [1]. Moreover, in the engineering problem to increase the stability of a rod, one also imposes a fixed interior point except for the ends of the rod [2]. Along this line, the solutions of multipoint boundary value problems for ordinary differential equations have great significance in mathematical theory and practical applications [3–5]. It is seen that the three-point boundary value problems for nonlinear second-order ordinary differential equations have attracted much attention [6–13]. On the other hand, it is significant to give the solutions and Green's functions of some three-point boundary value problems for linear second-order ordinary differential equations [9,14,15]. One can find that the solutions of linear three-point boundary value problems can be used in solving nonlinear ones [9,15].

As shown in the above mentioned works, the existence and uniqueness of the solution are always focused on by using a fixed point theorem. However, from the viewpoint of practical applications, one should give the explicit solutions or approximate solutions of multi-point boundary value problems. In the present paper, we will generally consider the following three-point boundary value problems for linear second-order ordinary differential equations with variable coefficients:

$$\varphi''(x) + p(x)\varphi'(x) + q(x)\varphi(x) = g(x), \quad x \in (a, b), \tag{1}$$

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http://dx.doi.org/10.1016/j.amc.2014.08.076
0096-3003/

with

$$\varphi(a) = \alpha, \quad \varphi(b) + \lambda\varphi(\xi) = \beta, \quad \xi \in (a, b), \tag{2}$$

or

$$\varphi(a) + \mu\varphi(\xi) = \alpha, \quad \varphi(b) = \beta, \quad \xi \in (a, b), \tag{3}$$

where the known functions $p(x) \in C^1[a, b]$ and $q(x),\ g(x) \in C[a, b]$. $\alpha,\ \beta,\ \lambda,\ \mu$ and ξ are the constants. By using the integration method, the boundary value problems (1) with (2) or (3) will be transformed to a Fredholm integral equation of the second kind. Then we gives sufficient conditions for the unique solution of the obtained Fredholm integral equations. A novel, simple and efficient method is proposed to give the approximate solutions of Fredholm integral equations of the second kind. Numerical examples are carried out to verify the proposed methods.

The paper is structured as follows. In Section 2, the integration method is applied to transform the boundary value problems (1) with (2) or (3) to a Fredholm integral equation of the second kind. The sufficient conditions for the unique solution of the given Fredholm integral equation are shown in $\mathbb{L}^2[a, b]$ by using the Banach fixed point theorem. Section 3 presents a new numerical method to solve Fredholm integral equations of the second kind, and the convergence together with error estimate of the approximate solution are made. Some numerical examples are calculated in Section 4 to show the effectiveness of the proposed methods. Some main conclusions are shown in Section 5.

2. Fredholm integral equations

Now let us transform the boundary value problems (1) with (2) or (3) to a Fredholm integral equation of the second kind. Then some special cases will be considered and their Green's functions will be derived again. The theorems for the unique solution of the obtained Fredholm integral equations will also be given.

2.1. Transformation method

It is convenient to define the function $V(x, t)$ as

$$V(x, t) = p(t) + (x - t)[q(t) - p'(t)] \tag{4}$$

and one has the following two theorems.

Theorem 1. *If* $(b - a) + \lambda(\xi - a) \neq 0$, *the boundary value problem*

$$\begin{cases} \varphi''(x) + p(x)\varphi'(x) + q(x)\varphi(x) = g(x), & x \in (a, b) \\ \varphi(a) = \alpha, \quad \varphi(b) + \lambda\varphi(\xi) = \beta, & \xi \in (a, b) \end{cases} \tag{5}$$

is equivalent to the following Fredholm integral equation of the second kind

$$\varphi(x) + \int_a^b K_1(x, t)\varphi(t)dt = h_1(x), \tag{6}$$

where

$$K_1(x, t) = \begin{cases} \frac{(b-x)+\lambda(\xi-x)}{(b-a)+\lambda(\xi-a)} V(a, t), & a \leqslant t \leqslant \min(x, \xi) \leqslant b, \\ \frac{(a-x)}{(b-a)+\lambda(\xi-a)} V(b, t), & a \leqslant \max(x, \xi) \leqslant t \leqslant b, \\ \frac{(b-x)V(a,t)+\lambda(\xi-a)V(x,t)}{(b-a)+\lambda(\xi-a)}, & a \leqslant \xi \leqslant t \leqslant x \leqslant b, \\ \frac{(a-x)[V(b,t)+\lambda V(\xi,t)]}{(b-a)+\lambda(\xi-a)}, & a \leqslant x \leqslant t \leqslant \xi \leqslant b, \end{cases} \tag{7}$$

$$h_1(x) = \int_a^x (x - t)g(t)dt + \frac{(a - x)}{(b - a) + \lambda(\xi - a)}\left[\int_a^b (b - t)g(t)dt + \lambda\int_a^\xi (\xi - t)g(t)dt\right] + \frac{(b - x) + \lambda(\xi - x)}{(b - a) + \lambda(\xi - a)}\alpha + \frac{(x - a)}{(b - a) + \lambda(\xi - a)}\beta. \tag{8}$$

Proof. We integrate both sides of the differential equation in (5) with respect to x from a to x twice and get

$$\varphi(x) + \int_a^x \{p(t) + (x - t)[q(t) - p'(t)]\}\varphi(t)dt = \int_a^x (x - t)g(t)dt + [\varphi'(a) + p(a)\varphi(a)](x - a) + \varphi(a). \tag{9}$$

In what follows, the boundary conditions in (5) are used to determine the unknowns $\varphi'(a)$ and $\varphi(a)$ respectively. It is assumed that $x = b$ in (9) and one has