Applied Mathematics and Computation 247 (2014) 47–60

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Two classes of implicit–explicit multistep methods for nonlinear stiff initial-value problems ☆

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ARTICLE INFO

Keywords:
Stiff problems
Singular perturbation problems
Implicit–explicit multistep methods
One-leg methods
Linear multistep methods
Convergence

ABSTRACT

The initial value problems of nonlinear ordinary differential equations which contain stiff and nonstiff terms often arise from many applications. In order to reduce the computation cost, implicit–explicit (IMEX) methods are often applied to these problems, i.e. the stiff and non-stiff terms are discretized by using implicit and explicit methods, respectively. In this paper, we mainly consider the nonlinear stiff initial-value problems satisfying the one-sided Lipschitz condition and a class of singularly perturbed initial-value problems, and present two classes of the IMEX multistep methods by combining implicit one-leg methods with explicit linear multistep methods and explicit one-leg methods, respectively. The order conditions and the convergence results of these methods are obtained. Some efficient methods are constructed. Some numerical examples are given to verify the validity of the obtained theoretical results.

1. Introduction

The initial-value problems (IVPs) of ordinary differential equations (ODEs) with stiff and nonstiff terms often arise from many science and engineering fields such as automatic control, circuits and semiconductor devices, atmospheric chemistry, fluid mechanics, astrophysics and spatial discretization of some initial boundary value problems of partial differential equations (see [1–5,8–11,15–17,22–33,35–42]). In order to reduce the computation cost, implicit–explicit (IMEX) methods are often applied to these stiff problems, and the stiff and non-stiff terms are discretized by using implicit methods and explicit methods, respectively.

Recently, IMEX Runge–Kutta methods, IMEX linear multistep methods and their deformations have attracted extensive attentions. Their order conditions, linear stability, strong stability, construction of efficient algorithms and their applications etc. are discussed (see [1–5,8–11,13–18,21–31,33–42] and the references therein). When high-order IMEX Runge–Kutta methods are applied to the problems with the stiff and non-stiff terms, the order reduction phenomenon may occur (see [8–11]).

One-leg methods and the corresponding linear multistep methods both include the famous BDF methods, and have the same linear stability properties and comparative computation cost. Compared with the corresponding implicit linear

☆ This research is supported by the National Natural Science Foundation of China (No. 11271311), the Chinese Program for Changjiang Scholars and Innovative Research Team in University (No. IRT1179) and the Research Foundation of Education Commission of Hunan Province of China (No. 14A146).

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http://dx.doi.org/10.1016/j.amc.2014.08.066
0096-3003/

multistep methods, implicit one-leg methods may have stronger nonlinear stability properties and better B-convergence property when they are applied to nonlinear stiff problems (see [12,32]). Therefore, we present two classes of IMEX multistep methods for nonlinear problems with stiff and non-stiff terms, where implicit one-leg methods are applied to discretize the stiff terms, and explicit linear multistep methods and explicit one-leg methods are applied to discretize the nonstiff terms, respectively.

The rest of the paper is organized as follows. In Section 2, we introduce two typical classes of stiff problems and the IMEX multistep methods combining implicit one-leg methods with explicit linear multistep methods. The order conditions of these methods are given, and some specific efficient IMEX multistep methods are constructed. In Section 3, we give the convergence results of the IMEX multistep methods for these classes of stiff problems. In Section 4, we present and discuss the IMEX one-leg methods for these stiff problem classes, which combine implicit one-leg methods with explicit one-leg methods. In Section 5, some numerical examples are given to verify the validity of the obtained theoretical results. In Section 6, we summarize the work in this paper.

2. Stiff problems and IMEX multistep methods

Firstly, we consider the IVPs of nonlinear stiff ODEs

$$\begin{cases} y'(t) = u(t, y(t)), & t \in [0, T], \\ y(0) = \eta, & \eta \in R^m, \end{cases} \tag{2.1}$$

where R^m denotes the m-dimensional Euclidean space, $u : D = [0, T] \times R^m \to R^m$ is a given sufficiently-smooth mapping and satisfies the one-sided *Lipschitz* condition

$$\langle u(t, y) - u(t, z), y - z \rangle \leqslant v\|y - z\|^2, \quad \forall (t, y), (t, z) \in D, \tag{2.2}$$

where v is the one-sided *Lipschitz* constant, $\langle \cdot, \cdot \rangle$ is the standard inner product on R^m and $\| \cdot \|$ is the corresponding norm induced by this inner product.

Applying linear k-step methods and the corresponding one-leg methods to the problems (2.1) yields

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j u(t_{n+j}, y_{n+j}), \tag{2.3}$$

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = hu\left(\sum_{j=0}^{k} \beta_j t_{n+j}, \sum_{j=0}^{k} \beta_j y_{n+j} \right), \tag{2.4}$$

where h is the given stepsize, $t_{n+j} = (n+j)h, \alpha_k \neq 0$, the generating polynomials $\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j$ and $\sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j$ satisfy $\rho(0) = 1$ and $\rho'(1) = \sigma(1) = 1, \rho(\xi)$ and $\sigma(\xi)$ have no common factors.

The linear k-step methods (2.3) and the corresponding one-leg methods (2.4) can be written as

$$\rho(E) y_n = h\sigma(E) u(t_n, y_n), \tag{2.5}$$

$$\rho(E) y_n = hu(\sigma(E) t_n, \sigma(E) y_n), \tag{2.6}$$

where E is the shift operator $Ey_n = y_{n+1}$. (2.5) and (2.6) include the famous BDF methods.

Now, we introduce the vector $Y_n = (y_{n+k-1}, y_{n+k-2}, \ldots, y_n)^T$ and its G-norm

$$\|Y_n\|_G^2 = \sum_{i=1}^{k} \sum_{j=1}^{k} g_{ij} \langle y_{n+i-1}, y_{n+j-1} \rangle,$$

where $G = (g_{ij})_{k \times k}$ is the given k-dimensional, real, symmetric and positive definite matrix.

Definition 2.1. [12,16]. The one-leg methods (2.4) (or (2.6)) are said to be G-stable, if there exists a k-dimensional, real, symmetric and positive definite matrix G such that two numerical solution sequences $\{y_n\}$ and $\{\hat{y}_n\}$ satisfy

$$\|Y_{n+1} - \hat{Y}_{n+1}\|_G \leqslant \|Y_n - \hat{Y}_n\|_G$$

for all stepsizes $h > 0$ and for the problems (2.1) and (2.2) with $v = 0$.

Due to the definition of G-stability, it is easy to show that G-stability implies A-stability for one-leg methods, and there exists the following relation between A-stability and G-stability.

Lemma 2.1. [12,16]. *The linear k-step methods* (2.3) (*or* (2.5)) *are A-stable if and only if the corresponding one-leg methods* (2.4) (*or* (2.6)) *are G-stable.*