Applied Mathematics and Computation 247 (2014) 90–99

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Solvability for fractional differential equations at resonance on the half line ☆

Weihua Jiang

College of Sciences, Hebei University of Science and Technology, Shijiazhuang 050018, Hebei, PR China

ARTICLE INFO

Keywords:
Fractional differential equation
Integral boundary conditions
Resonance
Fredholm operator
Coincidence degree theory

ABSTRACT

The existence of solutions for fractional differential equations with integral boundary conditions at resonance on the half line is investigated. Our analysis relies on constructing the suitable Banach space, defining appropriate projectors and the coincidence degree theory due to Mawhin. An example is given to illustrate our main result.

1. Introduction

In this paper, we study the following boundary value problems at resonance on the half line

$$\begin{cases} D_{0^+}^{\alpha}u(t) = f(t, u(t), D_{0^+}^{\alpha-2}u(t), D_{0^+}^{\alpha-1}u(t)), \quad \text{a.e. } t \in [0, +\infty), \\ u(0) = 0, \quad D_{0^+}^{\alpha-2}u(0) = 0, \quad D_{0^+}^{\alpha-1}u(+\infty) = \int_0^{+\infty} h(t) D_{0^+}^{\alpha-1}u(t)dt, \end{cases} \tag{1.1}$$

where $2 < \alpha \leqslant 3$, $D_{0^+}^{\alpha}$ is the standard Riemann–Liouville fractional derivative.

Recently, more and more authors pay their close attention to fractional differential equations because of its frequent appearance in a variety of different areas such as rheology, fluid flows, electrical networks, viscoelasticity, chemical physics, electron-analytical chemistry, biology and control theory etc. (see [1–27]). The boundary value problems of fractional differential equations at resonance on the finite interval have been investigated by many authors (see [19–23] and references cited therein). There are a few papers to investigate the boundary value problems of fractional differential equations at nonresonance on the half line (see [24–27]). For example, in [26], by using Schauder's fixed point theorem, the authors discussed the existence of unbounded solutions for the boundary value problem of fractional order at nonresonance on the half line:

$$\begin{cases} D_{0^+}^{\alpha}u(t) = f(t, u(t), D_{0^+}^{\alpha-1}u(t)), \quad 0 < t < +\infty, \\ u(0) = 0, \quad D_{0^+}^{\alpha-1}u(+\infty) = c, \end{cases}$$

where $c \in \mathbb{R}, 1 < \alpha \leqslant 2, f \in C([0, +\infty) \times \mathbb{R}^2, \mathbb{R})$. To the best of our knowledge, there are few papers to study the boundary value problems of fractional differential equations with integral boundary conditions at resonance on the half line. Motivated by the excellent results of [26,29], we will discuss about this problem by constructing the suitable Banach space, defining appropriate operators and the coincidence degree theory due to Mawhin.

☆ Supported by NNSF of China (No: 11171088) and NSF of Hebei Province (No: A2013208108).
E-mail address: weihuajiang@hebust.edu.cn

http://dx.doi.org/10.1016/j.amc.2014.08.067
0096-3003/

In this paper, we will always suppose that the following conditions hold.

(H_1) $2 < \alpha \leqslant 3, h \in L^1[0,+\infty)$, $\int_0^{+\infty} h(t)dt = 1$, there exists $\varphi(t) \in L^1[0,+\infty), \varphi(t) \neq 0, t \in [0,+\infty)$ such that

$$\int_0^{+\infty} \varphi(t) \int_0^t h(s)dsdt \neq 0.$$

(H_2) $f : [0,+\infty) \times \mathbb{R}^3 \to \mathbb{R}$ satisfies Carathéodory conditions, i.e. $f(\cdot,u)$ is measurable for each fixed $u \in \mathbb{R}^3$, $f(t,\cdot)$ is continuous for a.e. $t \in [0,+\infty)$, and for each $r > 0$, there exists $\Phi_r \in L^1[0,+\infty)$ such that $|f(t,x,y,z)| \leqslant \Phi_r(t)$ for all $\frac{|x|}{1+t^{\alpha-1}}, \frac{|y|}{1+t}, \ |z| \in [0,r], \ \text{a.e.} t \in [0,+\infty)$.

2. Preliminaries

For convenience, we introduce some notations and a theorem. For more details see [28].

Let X and Y be real Banach spaces and $L : dom(L) \subset X \to Y$ be a Fredholm operator with index zero, $P : X \to X, Q : Y \to Y$ be projectors such that

$$ImP = KerL, \quad KerQ = ImL, \quad X = KerL \oplus KerP, \quad Y = ImL \oplus ImQ.$$

It follows that

$$L|_{domL \bigcap KerP} : domL \bigcap KerP \to ImL$$

is invertible. We denote the inverse by K_P.

Assume Ω is an open bounded subset of $X, domL \bigcap \overline{\Omega} \neq \emptyset$. The map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact.

Theorem 2.1 [28]. *Let $L : domL \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(1) *$Lx \neq \lambda Nx$ for every $(x,\lambda) \in [(domL \setminus KerL) \bigcap \partial\Omega] \times (0,1)$;*
(2) *$Nx \notin ImL$ for every $x \in KerL \bigcap \partial\Omega$;*
(3) *$deg(QN|_{KerL}, \Omega \bigcap KerL, 0) \neq 0$, where $Q : Y \to Y$ is a projection such that $ImL = KerQ$.*

Then the equation $Lx = Nx$ has at least one solution in $domL \bigcap \overline{\Omega}$.

The following definitions and lemmas can be found in [1,2].

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$I_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds, \tag{2.1}$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2. The fractional derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$D_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} y(s)ds, \tag{2.2}$$

provided the right-hand side is pointwise defined on $(0,\infty)$, where $n = [\alpha] + 1$.

Lemma 2.1. *Assume $f \in L^1[0,+\infty), \ q \geqslant p \geqslant 0, \ q > 1$, then*

$$D_{0^+}^{p} I_{0^+}^{q} f(t) = I_{0^+}^{q-p} f(t).$$

Lemma 2.2. *Assume $\alpha > 0, \ \lambda > -1$, then*

$$D_{0^+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(n+\lambda-\alpha+1)} \frac{d^n}{dt^n} (t^{n+\lambda-\alpha}),$$

where n is the smallest integer greater than or equal to α.