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Discretization of fractional order differentiator over Paley–Wiener space

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ABSTRACT

The Paley–Wiener space consists of functions whose Fourier transform is compactly supported in the frequency domain. In the context of signal processing, such functions are also known as bandlimited signals, which represent a large class of signals in signal processing. An analog fractional order differentiator is representable by way of the Cauchy integral formula, special functions, as well as the Fourier/Laplace transformer, while a digital differentiator of fractional order can be obtained through direct or indirect discretization techniques. In this paper, we present the design of a finite impulse response (FIR) filter that discretizes the fractional order differentiator over functions in Paley–Wiener space. The proposed FIR model has some meritorious properties that are preferred in applications: the filter coefficients are independent of the signal samples; it is capable of interpolating or extrapolating at an arbitrary point in the sampling domain; it is adaptive to uniform or non-uniform sampling scenarios. We present explicit formulas on the matrices that lead to computing the filter coefficients. A closed form formula on the error of approximation is derived to demonstrate the accuracy of the proposed discretization model of fractional derivatives. Numerical results also show that the proposed method is more computationally efficient than the well-known methods such as the Grünwald–Letnikov approximation.

1. Introduction

The concept of fractional order differentiation and integration came into being at the same time as the conventional integral order calculus. The early development of fractional calculus is associated with the names such as Euler, Riemann, Leibniz, Lagrange, Fourier, Laplace, and so on. A thorough historical review of fractional calculus can be found in [1]. Fractional order differentiation or a fractional derivative is considered a generalization of an integral order derivative, however, with its own characteristics. Only in recent years have fractional derivatives become important mathematical tools in modeling various physical phenomena. Some also believe that fractional order differential equations are more effective in modeling nonlinearity. Podlubny gave celebrated geometrical and physical interpretation of fractional derivative in [2]. Meanwhile, the applications of the fractional derivative are readily seen in areas such as thermodynamics, electromagnetic field, quantum mechanics [3–5], signal processing, and control [6,7], and cardiac electrophysiology [8], to name a few.

In this paper, we are concerned with constructing a discrete analog of a fractional order differentiator over Paley–Wiener space, which consists of functions with compact support in the frequency domain. These types of functions are also known as bandlimited signals, representing an important class of signals in signal processing. In practice, all signals can be filtered into

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bandlimited signals, either low-passed or band-passed, based on the appropriate filters. Tseng, Pei and Hsia introduced their design of a digital FIR differentiator in [6] based on Cauchy integral formula and Fourier transform to discretize the fractional derivative kernel in the Fourier frequency domain. Their digital fractional differentiator (DFD) is capable of estimating the fractional derivative at the current sampling point (uniform sampling). The performance of their DFD is demonstrated with numerical experiments. The DFD proposed in this paper has the following novel and meritorious features: (i) it is capable of interpolating and extrapolating fractional derivatives at an arbitrary point in the sampling domain; (ii) it is adaptive to uniform or non-uniform sampling; (iii) it comprises the discretization of an integral order differentiator as a special case. We present a closed form bound on the error of approximation to prove the accuracy of the proposed model.

The rest of this paper is organized as follows. In Section 2, we give necessary background on fractional derivatives along with a list of formulas that motivate the proposed design, the fractional order differentiator and underlying function space. The main results are reported in Section 3, including derivations, analysis, and numerical tests, followed by conclusive remarks and potential applications of the proposed DFD.

2. Definitions

Unlike general integral order derivatives, $d^n f/dx^n$, where one can obtain higher order derivatives by repeating the process of first-order differentiation, the fractional order derivative, denoted by, D^α, where $\alpha \in \mathbb{R}^+$, is defined through a fractional integral,

$$D^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_\alpha^t (t-\tau)^{\alpha-1}f(\tau)d\tau$$

known as a Riemann–Liouville fractional integral. As such a fractional derivative is obtained as

$$D^\alpha f(t) = \frac{d^n}{dt^n}D^{-(n-\alpha)}f(t) \tag{1}$$

where $n = \lceil \alpha \rceil$. The more popular Caputo fractional derivative [9] is particularly useful for solving fractional differential equations,

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_\alpha^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau, \quad (n-1 \leqslant \alpha < n). \tag{2}$$

There are a few other alternative formulas for fractional order derivatives. The most relevant formula to this paper is the Grünwald–Letnikov derivative

$$D^\alpha f(t) = \lim_{h\to 0} h^{-\alpha}\sum_{j=0}^{\lfloor (t-\alpha)/h \rfloor}(-1)^j\binom{\alpha}{j}f(t-jh).$$

Suppose we only approximate the derivative at the mesh points, $t_j = t_0 + jh,\ j = 0, 1, \ldots, N$, the Grünwald–Letnikov approximation formula can be written as [10]

$$D^\alpha f(t_N) \doteq h^{-\alpha}\sum_{j=0}^{N}\omega_j^{(\alpha)}f(t_{N-j}) \tag{3}$$

where $\omega_0^{(\alpha)} = 1,\ \omega_j^{(\alpha)} = (1-(\alpha+1)/k)\omega_{j-1}^{(\alpha)},\ j = 1, 2, 3, \ldots N$. Eq. (3) is clearly a finite difference approximation of the fractional derivative of order α. There are higher-order finite difference methods for numerical differentiation of integral-order derivatives, such as the ones derived from Taylor series and those based on Richardson extrapolation. Consequently, those difference formulas are generalized to approximate fractional derivatives [10–12]. Lubich also developed higher order method by way of the numerical quadrature of fractional order in [13]. Moreover, in the same paper, he also laid groundwork in the convergence and stability analysis of numerical methods for discretized fractional calculus.

The proposed algorithm in this paper for a discrete fractional differentiator is motivated by (3). We observe from (3) that a fractional derivative is a process with memory. The derivative depends on the function's historical values. Unlike the integral-order derivatives, where only local function values are needed, in the case of a fractional derivative, it depends on function values over the whole domain and so is a nonlocal operator. This is also observed from the integral definitions of fractional derivatives in (1) and (2).

There are a number of known formulas, see [1], in addition to the above mentioned ones, for computing fractional order derivatives, none of which would readily produce closed form, explicit expressions for the derivative, except for some rather basic functions. This is due to the complexity of the definitions of fractional derivatives, and most of the differentiation rules for ordinary derivative cannot be carried over to the fractional derivative. Therefore, there is a need for developing computational methods that are capable of approximating fractional derivatives numerically with high accuracy, whilst the method itself has a simple structure for implementation. The model proposed in this paper meets this criterion since it is formulated in the form of a convolution between a set of optimal coefficients and a set of functional samples, and it is capable of