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Solitary wave solutions for nonlinear partial differential equations that contain monomials of odd and even grades with respect to participating derivatives

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ABSTRACT

We apply the method of simplest equation for obtaining exact solitary traveling-wave solutions of nonlinear partial differential equations that contain monomials of odd and even grade with respect to participating derivatives. We consider first the general case of presence of monomials of the both (odd and even) grades and then turn to the two particular cases of nonlinear equations that contain only monomials of odd grade or only monomials of even grade. The methodology is illustrated by numerous examples.

1. Introduction

Traveling waves exist in many natural systems. Because of this traveling wave solutions of the nonlinear partial differential equations are studied much in the last decades [1–5] and effective methods for obtaining such solutions for integrable systems are developed [6,7]. Our discussion below will be based on a specific approach for obtaining exact special solutions of nonlinear PDEs: the method of simplest equation and its version called modified method of simplest equation [8–11]. The method of simplest equation is based on a procedure analogous to the first step of the test for the Painleve property [12–14]. In the version of the method called modified method of the simplest equation [10,11] this procedure is substituted by the concept for the balance equation. This version of the method of simplest equation has been successfully applied for obtaining exact traveling wave solutions of numerous nonlinear PDEs such as versions of generalized Kuramoto–Sivashinsky equation, reaction–diffusion equation, reaction-telegraph equation [10,15] generalized Swift–Hohenberg equation and generalized Rayleigh equation [11], extended Korteweg–de Vries equation [16], etc. [17,18].

The method of simplest equation works as follows. By means of an appropriate ansatz the solved nonlinear PDE is reduced to a nonlinear ODE

$$P(u, u_\xi, u_{\xi\xi}, \dots) = 0. \tag{1}$$

Then the finite-series solution $u(\xi) = \sum_{\mu=-\nu}^{\nu_1} p_\mu [f(\xi)]^\mu$ is substituted in (1). p_μ are coefficients and $f(\xi)$ is solution of simpler ordinary differential equation called simplest equation. Let the result of this substitution be a polynomial of $f(\xi)$. $u(\xi)$ is a solution of Eq. (1) if all coefficients of the obtained polynomial of $f(\xi)$ are equal to 0. This condition leads to a system of nonlinear algebraic equations. Each non-trivial solution of this system corresponds to a solution of the studied nonlinear PDE.

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In the last years we observe a fast development of tools that are connected to the method of simplest equation. These tools are effectively used for obtaining exact solutions of many nonlinear differential equations. As an example we mention [19] where in a systematic way on the basis of the Q-expansion method of Kudryashov and by using the concepts of ordinary and differential monomials and Newton polygons the authors obtained the forms of many nonlinear differential equations that have given kind of solution. Similar idea will be used below.

We shall consider traveling-wave solutions $u(x,t) = u(\xi) = u(\alpha x + \beta t)$ constructed on the basis of the simplest equation

$$f_\xi^2 = 4(f^2 - f^3), \tag{2}$$

which solution is $f(\xi) = \frac{1}{\cosh^2(\xi)}$. α and β are parameters. Let us note here that the transformation $f(\xi) = 1 - u(\xi)$ transforms Eq. (2) to the equation $u_\xi = \pm(1 - u^2)$. The last equation is a version of the Riccati equation which is much used as a simplest equation in the applications of the method of simplest equation [20,21].

In addition we shall use the following concept of grade of monomial with respect to participating derivatives. Let us consider polynomials that are linear combination of monomials where each monomial contains product of terms consisting of powers of derivatives of different orders. This product of terms can be multiplied by a polynomial of u. Let a term from a monomial contain kth power of a derivative of lth order. We shall call the product kl grade of the term with respect to participating derivatives. The sum of these grades of all terms of a monomial will be called grade of the monomial with respect to participating derivatives. The general case is **(1):** The polynomial contains monomials that contain derivatives have odd or even grades with respect to participating derivatives. There are two particular cases: **(1A):** All monomials that contain derivatives have odd grades with respect to participating derivatives; **(1B):** All monomials that contain derivatives have even grades with respect to participating derivatives. We shall formulate our main result for the general case **(1)** and then we shall demonstrate several applications for the cases **(1), (1A), (1B)**.

2. Main result

Below we search for solitary wave solutions for the class of nonlinear PDEs that contain monomials of derivatives which order with respect to participating derivatives is even and monomials of derivatives which order with respect to participating derivatives is odd.

Theorem. *Let $\mathcal{P}$ be a polynomial of the function $u(x,t)$ and its derivatives. $u(x,t)$ belongs to the differentiability class C^k, where k is the highest order of derivative participating in $\mathcal{P}$. $\mathcal{P}$ can contain some or all of the following parts:* **(A)** *polynomial of u;* **(B)** *monomials that contain derivatives of u with respect to x and/or products of such derivatives. Each such monomial can be multiplied by a polynomial of u;* **(C)** *monomials that contain derivatives of u with respect to t and/or products of such derivatives. Each such monomial can be multiplied by a polynomial of u;* **(D)** *monomials that contain mixed derivatives of u with respect to x and t and/or products of such derivatives. Each such monomial can be multiplied by a polynomial of u;* **(E)** *monomials that contain products of derivatives of u with respect to x and derivatives of u with respect to t. Each such monomial can be multiplied by a polynomial of u;* **(F)** *monomials that contain products of derivatives of u with respect to x and mixed derivatives of u with respect to x and t. Each such monomial can be multiplied by a polynomial of u;* **(G)** *monomials that contain products of derivatives of u with respect to t and mixed derivatives of u with respect to x and t. Each such monomial can be multiplied by a polynomial of u;* **(H)** *monomials that contain products of derivatives of u with respect to x, derivatives of u with respect to t and mixed derivatives of u with respect to x and t. Each such monomial can be multiplied by a polynomial of u.*

Let us consider the nonlinear partial differential equation

$$\mathcal{P} = 0. \tag{3}$$

We search for solutions of this equation of the kind $u(\xi) = u(\alpha x + \beta t) = u(\xi) = \gamma f(\xi)$, where γ is a parameter and $f(\xi)$ is solution of the simplest equation $f_\xi^2 = 4(f^2 - f^3)$. The substitution of this solution in Eq. (3) leads to a relationship $\mathcal{R}$ of the kind

$$\mathcal{R} = \sum_{i=0}^{N} C_i f(\xi)^i + f_\xi \left(\sum_{j=0}^{M} D_j f(\xi)^j \right) \tag{4}$$

where N and M are natural numbers depending on the form of the polynomial $\mathcal{P}$. The coefficients C_i and D_j depend on the parameters of Eq. (3) and on α, β and γ. Then each nontrivial solution of the nonlinear algebraic system

$$C_i = 0, \quad i = 1, \ldots, N; \quad D_j = 0, \quad j = 1, \ldots, M \tag{5}$$

leads to solitary wave solution of the nonlinear partial differential Eq. (3).

Proof. First we shall prove the following:

Let $f(\xi)$ be a solution of the nonlinear ordinary differential equation $f_\xi^2 = 4(f^2 - f^3)$. Then the even derivatives of $f(\xi)$ contain only a polynomial of $f(\xi)$. The odd derivatives of $f(\xi)$ contain a polynomial of $f(\xi)$ multiplied by f_ξ.