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Generating functions for the generalized Gauss hypergeometric functions

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ABSTRACT

Formulas and identities involving many well-known special functions (such as the Gamma and Beta functions, the Gauss hypergeometric function, and so on) play important rôles in themselves and in their diverse applications. Various families of generating functions have been established by a number of authors in many different ways. In this paper, we aim at establishing some (presumably new) generating functions for the generalized Gauss type hypergeometric type function $F_p^{(\alpha,\beta;\kappa,\mu)}(a,b;c;z)$ which is introduced here. We also present some special cases of the main results of this paper.

1. Introduction

Many important functions in applied sciences (which are popularly known as special functions) are defined via improper integrals or infinite series (or infinite products). During last four decades or so, several interesting and useful extensions of many of the familiar special functions (such as the Gamma and Beta functions, the Gauss hypergeometric function, and so on) have been considered by several authors (see, for example, [4–7]; see also the very recent work [2]). The above-mentioned works have largely motivated our present study.

Recently, Parmer [8] introduced and studied some fundamental properties and characteristics of the generalized Beta type function $B_p^{(\alpha,\beta;\mu)}(x,y)$ defined by Parmar (see [8, p. 37, Eq. (19)]):

$$B_p^{(\alpha,\beta;\mu)}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{t^{\mu}(1-t)^{\mu}}\right)dt, \tag{1.1}$$

$$(\Re(p)\geqq 0;\ \min\{\Re(x),\Re(y),\Re(\alpha),\Re(\beta)\}>0;\ \Re(\mu)>0),$$

which, in the special case when $\mu=1$, reduces immediately to the generalized Beta type function defined earlier as follows (see, for details, [7, p. 4602, Eq. (4)]; see also [6, p. 32, Chapter 4]):

$$B_p^{(\alpha,\beta)}(x,y) = B_p^{(\alpha,\beta;1)}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt, \tag{1.2}$$

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$$(\Re(p) \geqq 0;\ \min\{\Re(x), \Re(y), \Re(\alpha), \Re(\beta)\} > 0).$$

For $\alpha = \beta$, (1.2) reduces obviously to the extended Beta type function $B_p(x, y)$ due to [4] is defined by Chaudhry et al. (see [4, p. 20, Eq. (1.7)]; see also [5, p. 591, Eq. (1.7)]):

$$B_p(x, y) = B_p^{(\alpha,\alpha)}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left(-\frac{p}{t(1-t)}\right) dt \quad (\Re(p) \geqq 0). \tag{1.3}$$

Clearly, for the classical Beta function $B(x, y)$, we have the following relationships:

$$B(x, y) = B_0(x, y) = B_0^{(\alpha,\beta)}(x, y) = B_0^{(\alpha,\beta;1)}(x, y),$$

where

$$B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt \quad (\Re(x) > 0;\ \Re(y) > 0). \tag{1.4}$$

In the year 2004, by making use of the extended Beta function $B_p(x, y)$ defined by (1.3), Chaudhry et al. [5] extended the Gauss hypergeometric function ${}_2F_1$ as follows (see [5, p. 591, Eqs. (2.1) and (2.2)]):

$$F_p(a, b; c; z) := \sum_{n=0}^{\infty} (a)_n \frac{B_p(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \tag{1.5}$$

$$(|z| < 1;\ \Re(c) > \Re(b) > 0;\ \Re(p) \geqq 0),$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [15, p. 2 and pp. 4–6]; see also [14, p. 2]):

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1, 2, 3, \cdots\}), \end{cases} \tag{1.6}$$

provided that the Gamma quotient exists (see, for details, [16, *et seq.*] and [17, p. 22 *et seq.*]). Similarly, by appealing to the definition (1.2) of the generalized Beta function $B_p^{(\alpha,\alpha)}(x, y)$, Özergin [6] and Özergin et al. [7] introduced and studied a further potentially useful extension of the generalized Gauss hypergeometric functions as follows (see, for example, [7, p. 4606, Section 3]; see also [6, p. 39, Chapter 4]):

$$F_p^{(\alpha,\beta)}(a, b; c; z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta)}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \tag{1.7}$$

$$(|z| < 1;\ \min\{\Re(\alpha), \Re(\beta)\} > 0;\ \Re(c) > \Re(b) > 0;\ \Re(p) \geqq 0),$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined by (1.6).

Based upon the generalized Beta function in (1.1), Parmar [8] introduced and studied a family of the generalized Gauss hypergeometric functions defined by Parmar (see [8, p. 44])

$$F_p^{(\alpha,\beta;\mu)}(a, b; c; z) := \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta;\mu)}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \tag{1.8}$$

$$(|z| < 1;\ \min\{\Re(\alpha), \Re(\beta), \Re(\mu)\} > 0;\ \Re(c) > \Re(b) > 0;\ \Re(p) \geqq 0).$$

Clearly, we have

$$F_p^{(\alpha,\beta;1)}(a, b; c; z) = F_p^{(\alpha,\beta)}(a, b; c; z), \tag{1.9}$$

$$F_p^{(\alpha,\alpha;1)}(a, b; c; z) = F_p(a, b; c; z) \tag{1.10}$$

and

$$F_0^{(\alpha,\alpha;1)}(a, b; c; z) = {}_2F_1(a, b; c; z) \tag{1.11}$$

in terms of the familiar Gauss hypergeometric function ${}_2F_1$.

In several areas in applied mathematics and mathematical physics, generating functions play an important rôle in the investigation of various useful properties of the sequences which they generate. They are used to find certain properties and formulas for numbers and polynomials in a wide variety of research subjects such as, for example, modern combinatorics (see [1,3,17,18]). Agarwal et al. [2] gave some interesting new classes of generating functions involving the generalized Gauss type hypergeometric function $F_p^{(\alpha,\beta)}$ defined by (1.5). In the present sequel to the aforementioned and many other recent investigations (see, for example, [3,9–13,18–24]; see also the monograph on the subject of generating functions by Srivastava and Manocha [17]), we present some (presumably new) generating functions involving the following family of generalized Gauss type hypergeometric functions: